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Stress concentration in nonlinear viscoelastic composites

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Abstract. *A nonlinear viscoelastic problem of the mechanics of composites is solved within the framework of a second-order nonlinear theory. A viscoelastic functional is used to construct general defining relations. A stochastic boundary value problem for determining the stress concentration and its relaxation in polymer composite materials (PCM) is solved. To derive the complete system of second-order viscoelastic equations, the method of successive approximation is used. A generalization of the correspondence principle to nonlinear viscoelastic media is obtained. The relaxation functions averaged over the viscoelastic matrix and elastic inclusions and the stress concentration parameters are determined. Examples are given showing the importance of the mutual influence of nonlinear elastic and viscous properties of the components on stress redistribution near inclusions in multicomponent PCMs. As a practical result, one can note the possibility of predicting the long-term strength of a material when a viscoelastic stress field is known near inclusions.*

Keywords: *viscoelastic composite; nonlinear deformation; hereditary kernel; identification; computer modeling*

Introduction. Computer modeling of the overall, smeared properties of composite viscoelastic materials is the problem of a practical importance [1, 2, 3]. Its solutions being the evaluation of stress concentration in microstructure elements and the formulation of required criteria of durability which would correspond to classical methods of strength theory [4, 5]. In the hereditary theory of viscoelasticity, the mechanical properties of the medium are given by elastic constants, creep kernels, and relaxation kernel [5, 6, 10]. The problem of identifying the creep and relaxation kernels, establishing the connection between kernels and determining the parameters of kernel is one of the main problems of the theory of viscoelasticity [6, 8, 9]. In the case of a uniaxial stressed state, the heredity kernels and the parameters of the nuclei are determined directly from the results of approximating the direct measurements of deformations or stresses in the process of creep or relaxation by functions that define the kernel. A detailed analysis of the methods for selecting the kernel structure and methods for determining the rheological parameters of linearly viscoelastic materials under a uniaxial stressed state is presented in [6, 9, 10]. The task of identifying heredity kernels in a complex stress state is more complex and reduces to establishing the relationship between the heredity kernel under a complex and uniaxial stress state. The uniaxial stress state is realized directly in the experiment and is considered as the base one. But as to composite material, a relationships would be established between the shear and volume creep kernels, as well as the longitudinal and transverse creep kernel. A dependence between the creep kernels in the complex stress state of multicomponent composite and local stress concentration are very important problems from the point of view of long-term strength prediction.

Research objective. We consider here the overall response and creep behavior of a random multi-component composites with nonlinear constituents. It was shown in [6] that for a nonlinear model described by means of approximations with third-order splines, the limiting value of the small deformation at which the deformation takes place linearly is 0.01. In the range [0,01; 0,09] we use the theory of small but nonlinear deformation. The shapes of the initial undeformed and deformed body are approximately the same. In the framework of the Rabotnov's type quasi-linear theory, the response of the viscoelastic material at time t is described by a linear law relating the stress $\sigma(t)$ with the elastic response σ^e as follows [5]:

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$$\sigma^{e}(t)=\int_{-\infty}^{t} g(t-t_1)\frac{\partial}{\partial t_1}\sigma(t_1)dt_1\,; \qquad \sigma^{e}(t)=2\frac{\partial}{\partial C}W(C,t); \qquad C=F^T F. \tag{1}$$

Here σ - the elastic second Piola–Kirchhoff stress tensor, coincides with Cauchy stress in the case of small deformation [7], $W(\mathbf{C},t)$ - instantaneous elastic potential, $\boldsymbol{C}$ - right Cauchy deformation tensor, $F=1+H$ - deformation gradient, $H=\frac{\partial u}{\partial x}$ - displacement gradient, $g(t)$ - isotropic reduced hereditary compliance tensor. It should be noted that the function $\sigma^{e}(t)$ here plays the role of the strain in the conventional theory of viscoelasticity. We use here Stieltjes convolutions which ideally suited to the study of composites [4, 9]. Equations (1) is quasi-linear because $\sigma^{e}(t)$ is nonlinear in the deformation tensor $\boldsymbol{C}(t)$, but the convolution operator is linear.

While the general theory is precisely that given in [8], the component form of the tensor equations is here given in terms of the components of the compliance tensor. This choice is dictated by the fact that we are mainly interested in the creep behavior of composites [6, 10]. Of course, if the creep compliance functions are known, then the stress relaxation functions can be obtained by the usual procedure [3, 6, 9]. We assume that function $W(\boldsymbol{C})$ is known in each of volumes a viscoelastic material with the properties governed by the stored energy function of third order as to displacement gradient. So the stored energy in the viscoelastic material may be expressed according to the Staverman-Schwarzl formula as follows

$$W(\eta,t)=\frac{1}{2}\int_{0^-}^{t}\int_{0^-}^{t} E\left(2t-t_1-t_2\right)d\eta\left(t_1\right)d\eta\left(t_2\right)+\frac{1}{3}\int_{0^-}^{t}\int_{0^-}^{t}\int_{0^-}^{t} G\left(3t-t_1-t_2-t_3\right)d\eta\left(t_1\right)d\eta\left(t_2\right)d\eta\left(t_3\right). \tag{2}$$

$$2\eta=C-1=H+H^T+H^T H=e+H^T H.$$

$E(t)$ - four order linear relaxation tensor, $G(t)$ - six order isotropic nonlinear relaxation tensor [2]. Second order approximation gives such a relations

$$E_{ijkl}=\mu(\alpha_1\delta_{ij}\delta_{kl}+2I_{ijkl}); \quad \eta_{ij(2)}=e_{ij(2)}+\frac{1}{2}(H_{mi}H_{mj})_{(1)};$$

$$G_{ijklmn}=\frac{1}{2}\nu_1\delta_{ij}\delta_{kl}\delta_{mn}+\nu_2(\delta_{ij}I_{klmn}+\delta_{kl}I_{ijmn}+\delta_{mn}I_{ijkl})+4\nu_3 I_{ijklmn};$$

$$I_{ijklmn}=\frac{1}{2}(I_{ipkl}I_{jpmn}+I_{jpkl}I_{ipmn}), \qquad I_{ijkl}+\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}); \tag{3}$$

The subscript in parenthesis stays for the order of approximation of nonlinear displacement, δ_{ij} is the Kronecker delta, $E(t)$ - is relaxation function of viscoelasticity or elastic modules tensor in the case of linear elastic strain theory. Function $G(t)$ is relaxation function of nonlinear viscoelasticity or Lame third order constants tensor in pure elastic case. Using the technique described in the previous works [5,8] the solution can be written as the convolution type integral over $B\cap t$ domain

$$e_{(1)}\left(x_1,t\right)=\Gamma\left(x_1,x_2,t,t'\right)*\tau_{(1)}\left(x_2,t'\right), \tag{4}$$

$\Gamma\left(x_1,x_2,t,t'\right)$ - is an operator with the kernel expressed through derivatives of the Green's function $u^{*}\left(x_1,x_2,t,t'\right)$.

Take now multi-phase isotropic material with the viscoelastic matrix being reinforced by randomly oriented in space inclusions of ellipsoid form. The result of $\Gamma(x,y,t,t')$ convolution with any two rank tensor function $b(y,t')$ may be obtained by integral

$$(\Gamma * b)_{ij}=\int_B u^{*}_{a(i,j)b}(x-y)b(y,t')dy+\oint_{\partial B} u^{*}_{a(i,j)}(x-y,t')bn_b dy\,. \tag{5}$$

With boundary condition $b(y,t)=b^0$, $\forall\, y\in\partial B$ it transformed to more simple relation

$$(\Gamma * b)_{ij}=\int_B u^{*}_{a(i,j)b}(x-y,t,t')[b(y,t')-b^0]dy. \tag{6}$$

After differentiation (2) the viscoelastic strain-stress relations may be written as [8]

$$e(t)=\int_0^t (J(t-t_1)+D(s,t-t_1))d\sigma(t_1), \qquad \sigma(t)=\int_0^t (E(t-t_1)-G(p,t-t_1))de(t_1), \tag{7}$$

$J(t), D(s,t)$, and $E(t), G(p,t)$ are the creep compliances and stress relaxation stiffness tensors, respectively, $s(J_2, J_3)$ and $p(I_2, I_3)$ - scalar functions of stress and strain invariants. For example here I_i - main invariants of the nonlinear Green deformation tensor η, i.e.

$$I_1 = tr(\eta); \qquad I_2 = \frac{1}{2}\left(I_1^2 - tr(\eta^2)\right); \qquad I_3 = \frac{1}{3}\left(tr(\eta^3) - I_1^3 - 3I_1 I_2\right);$$

$$\eta = \frac{1}{2}(F^T F - 1); \qquad F = \partial x / \partial X = 1 + H; \qquad x(t) = X + u(X,t), \tag{8}$$

X - coordinates of initial state, $\boldsymbol{x}(t)$ -coordinates of current, deformed state.

Basic maintenance and results of research. The complete analogy between the general theory of elasticity and the properties of a viscoelastic medium follows immediately from the theory of viscoelastic strain-stress relations derived Biot [4] in the context of thermodynamics. It was shown that the mechanics of viscoelastic media may be obtained from the theory of elasticity by the simple rule of replacing the elastic coefficients by operators. We realize here a similar applications and extend the correspondence principle to a quasi-linear viscoelastic medium with constitutive equation (1). If $E(t), G(p,t)$ and $J(t), D(s,t)$ are smooth functions of t variable, then applying the Carson transform [9]

$$C\{f(t)\} = f^*(z) = z\int_0^\infty e^{-zt} f(t)dt \tag{9}$$

to (3) gives

$$s^*(z) = (E^*(z) - G^*(p,z))e^*(z), \quad e^*(z) = (J^*(z) + D^*(s,z))s^*(z), \tag{10}$$

the star indicates the transformed function in the Carson domain, and z is the transform variable. Nonlinear response of composites may be in principle described as disturbance of linear problem, linearization or expansion in series [5, 7]. Known solution of the equations of displacement for the n step, we substitute it into the equations for the $n+1$ step, and so on. In second order theory of viscoelasticity we have the sequence

$$u^{(2)}(t) = \varepsilon u_1(t) + \varepsilon^2 u_2(t), \qquad u^{(2)}(t) = u^{(1)}(t) + \varepsilon u_2(t). \tag{11}$$

Here $u^{(1)}(t)$- solution of the linear viscoelasticity problem [6]. As to quasi-linear approach (1) we deal with constitutive equations for statistical fluctuations of first and second order displacement, deformation and stress in the reference representative volume written in the form

$$\sigma^e(t) = \int_{-\infty}^t g(t-t_1)d\sigma^{(1)}(t_1) - \sigma^{e(2)}(t); \qquad \sigma^{e(2)}(t) = Ge_{(1)}^2(t). \tag{12}$$

The representative volume of composite material we consider here as an infinite homogeneous viscoelastic solid which contains a set of disoriented ellipsoidal inclusions. We require $\sigma(t) = 0$ for $t < 0$. Subsequently the reference configuration of the inclusion material may spontaneously jump at $t = 0$ and thereafter is smoothly changing in such a way that the new reference configuration is obtained by giving the old reference configuration a not known strain $e(t)$. We solve this problem by using an extension of method [5, 8] to nonlinear viscoelastic deformation. By taking the Carson transforms we can reduce (6), (7), (12) to the corresponding elastic problem. Thus following [9], we obtain a representation theorem for the Carson transform of the Green's function in (6). Upon application of the Carson transform, the boundary value problem for the local stress and strain fields in matrix and inclusions becomes like a linear elastic problem in the Carson domain. Then the method proposed in [1, 3] can be applied to construct the model of effective behavior of the composite. Hence, it follows from (1) and (7) that

$$\tilde{J}^*(z) = \tilde{J}^*(J^*_r(z), J^*_r(z), c_i), \qquad \tilde{D}^*(z) = \tilde{D}^*(J^*_r(z), D^*_r(s_r, z), c_i). \tag{13}$$

Note that the constant c_i -volume concentration of phases remains unchanged after transforming from the time domain to the Carson domain. This distinguishes the Carson transform from the Laplace transform.

$$\tilde{J}(t) = (LC)^{-1}\tilde{J}^*(z) = \frac{1}{2\pi i}\int_{\delta - i\infty}^{\delta + i\infty} \frac{e^{zt}}{z}\tilde{J}^*(z)dz. \tag{14}$$

When the expressions for $\tilde{J}^*(z)$, $\tilde{D}^*(p,z)$ are complicated, it is difficult to evaluate analytically the integrals given in (9). Accordingly, a suitable numerical method is usually needed. There exist efficient algorithms for numerically evaluating the inverse Laplace transform. We use here the Fortran90 program from NAG-Fortran library. Statistical averaging of expression performed to define the mean deformation of anisotropic inclusions randomly oriented in volume. The result is overall response of such a composite isotropic one.

$$< e(t) >= \tilde{J}(0)\int_{-\infty}^t g(t-t_1)d < \sigma > + \tilde{J}(0) < \sigma^{(2)}(t) >. \tag{15}$$

As an example write here the expressions for second order Lame operators $\nu_k(t)$, that connected with third invariant of deformation $I_3(\eta)$, in Carson domain

$$\tilde{\nu}_1^*(z) = \sum_{r=1}^{n+1} 3c_r \begin{bmatrix} l_A\left[2\lambda\kappa_A\left(3\kappa+\mu\right)_A + n_A f_A\right] + 9\nu_1\kappa_A^3 + \\ 6\nu_2 l_A \kappa_A \left(3\kappa+2\mu\right)_A + 8\nu_3 l_A^2 n_A \end{bmatrix}_1^* (z);$$

$$\tilde{\nu}_2^*(z) = \sum_{r=1}^{n+1} c_r \left[4\mu_A^2\left(\frac{1}{2}f_A + 3\nu_2\kappa_A + 4\nu_3 l_A\right) - \frac{1}{2}\left(3\lambda\kappa_A + 2\mu l_A\right)\right]_r^* (z);$$

$$\tilde{\nu}_3^*(z) = \sum_{r=1}^{n+1} c_r \left[\frac{3}{2}\mu\mu_A\left(4\mu_A^2 - 1\right) + 8\nu_3\mu_A^3\right]_r^* (z). \qquad (16)$$

Here we denote $\nu_{1r}^*(z),\ \nu_{2r}^*(z),\ \nu_{3r}^*(z)$ are the relaxation functions of third order of r-component in Carson domain, Taking into account (11) the constitutive equation for isotropic composite material may be written in form

$$\langle\sigma(z)\rangle = \tilde{\mu}(z)\left[2\langle\eta\rangle + \left(\alpha_1 I_1(\langle\eta\rangle) + \alpha_3 I_1^2(\langle\eta\rangle) + \alpha_4 I_2(\langle\eta\rangle)\right)1 + \alpha_5 I_1(\langle\eta\rangle)\langle e\rangle + \alpha_6\langle e\rangle^2\right]; \qquad (17)$$

Parameters α_r are connected with determined in (11) ν_r, for example $\alpha_6 = 4(1+\nu_3/\mu)$ [8].

We consider boron and SIC-glass (elastic) spheres in epoxy resin matrix. The experimental data for the polymer is taken from the paper of [10]. This particular polymer is rather useful for test purposes since its behavior at long times is markedly different from its behavior at short time. Rheological parameters of epoxy resin were determined from experimental data [10] with method proposed earlier in [6]. Stress concentration near inclusions and overall creep response are modeled in the three-component composite with epoxy resin viscoelastic matrix. Instant nonlinear elastic properties of phases are in Table 1.

Table 1

Constituent nonlinear elastic material constants, GPa, for the B/SiC/Epoxy composite

Material	E	ν	ν_1	ν_2	ν_3
Boron	467.3	0.361	-840.0	-420.0	-390.0
SiC	440.3	0.171	-227.2	31.5	-170.75
Epoxy resin	3.15	0.382	13.3	4.09	-10.02

The solution of the system (10) is obtained by means of the Carson transform, where the kernel of shear relaxation are the fractional exponential functions of Rabotnov

$$\xi R_\alpha(t) = -\xi\frac{d}{dt}E_m\left(-\beta t^m\right), \quad m = 1+\alpha. \qquad (18)$$

Here $E_m(t)$ is the Mittag-Leffler function–

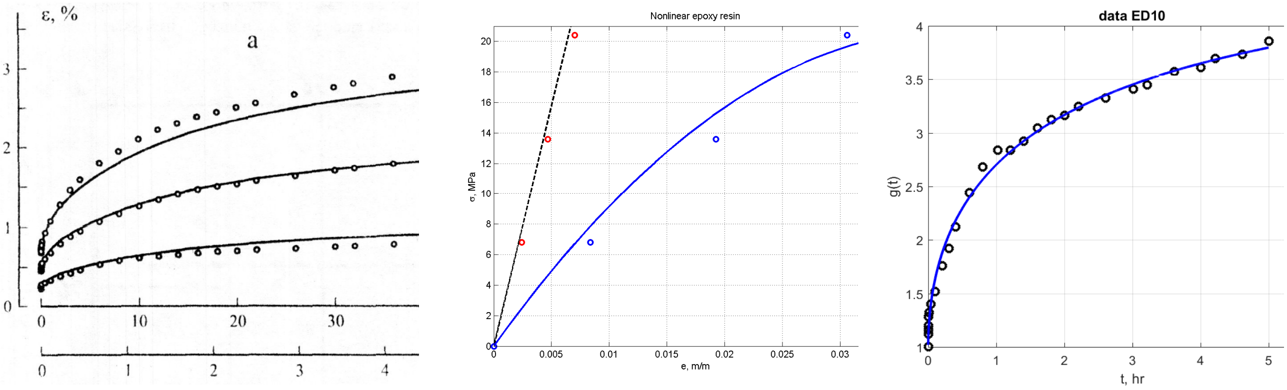
$$E_m(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(1+mn)}. \qquad (19)$$

Experimental creep data from [10] were used for identification rheological parameters of epoxy resin. Using methodology presented in [6] we obtain

$$\xi = 0.0189, hr^{-m}; \quad \beta = 0.052, hr^{-m}; \ \alpha = -0.4790. \qquad (20)$$

Figure 1.a shows the original experimental results of epoxy resin creep under uniaxial tension at different stress levels. From the analysis of which, in particular, the conclusion follows that the matrix is not linearly deformed, and, thus, the necessity of attracting a nonlinear theory of viscoelasticity. In Fig. 1.b shows the instantaneous curve of the nonlinear dependence of stress strain obtained as a result of processing isochronous creep curves. The dashed curve corresponds to the linear deformation variant. Circles indicate experimentally measured points. As we can see, the response of the epoxy resin is essentially nonlinear even within the limits of small but nonlinear deformations. In Fig. 1.c shows the calculated creep curve, constructed using the results of identification of the rheological parameters of the Rabotnov kernel (18). In Fig. 2 shows the calculated creep curve, constructed using the results of identifying the rheological parameters of the Rabotnov kernel. The circles indicate the experimental values. It can be noted that the proposed algorithm for determining the viscoelastic parameters of the binder can well be used for further analysis of the composite.

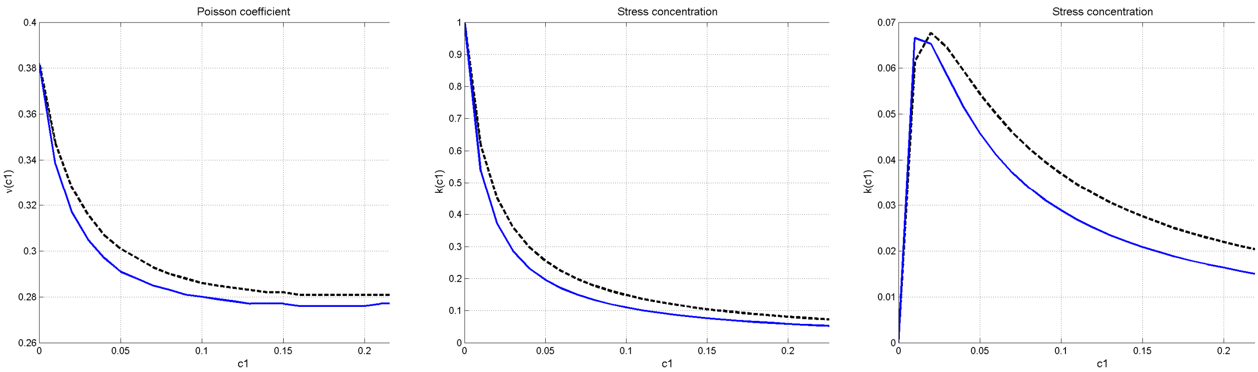
In Fig. 2.a shows the curves that illustrate the dependence of the overall nonlinear second-order relaxation function $\tilde{\nu}_3(c_1)$ on the filler concentration c_1. The dashed curve corresponds to the instantly elastic response of the material. The solid curve describes the corresponding dependence at the time moment of the order of 10000 hours. In



a *b* *c*

Fig. 1. Long term nonlinear creep data (a) and creep data for epoxy resin ED 10 (b) and nonlinear stress-strain elastic response (c)

Fig. 2.b shows the curves of the stress concentration coefficient $k_{11}(c_1,\langle\sigma_{11}\rangle)$ in the matrix material under uniaxial tension. The dashed curve corresponds to the instantly elastic state of the material. The solid curve describes the corresponding dependence at the time moment of the order of 10000 hours. In Fig. 2.c, as an illustration of possible nonlinear effects, the dependence of the shear stress concentration coefficient $k_{12}(c_1,\langle\sigma_{11}\rangle)$ in the matrix material under uniaxial tension is given. As before, the dashed curve corresponds to the instantly elastic state of the material. The solid curve describes the corresponding dependence at the time moment of the order of 10000 hours.



a *b* *c*

Fig. 2. Elastic and longterm nonlinear relaxation (a) stress concentration in epoxy resin matrix (b) and shear stress concentration in matrix, elastic and longterm response (c)

Summary

Increasing the reliability of composite structures is inevitably associated with the development of new modern models of strain and redistribution of stresses in microstructure elements. To take into account possible nonlinear effects during long-term operation, a nonlinear viscoelastic problem of the mechanics of composites is considered in the framework of a second-order nonlinear theory. A viscoelastic functional is used to construct general defining relations. The behavior of viscoelastic matrix is described using physically and geometrically nonlinear theories. Physical nonlinearity corresponds to the situation when the region of linear behavior is actually not observed even for small but nonlinear deformations. These two effects may be taken into account simultaneously in solving some problems of creep mechanics of polymers and PCM. We use here the most convenient quasi-linear model of the Rabotnov type. Examples are given showing the importance of the nonlinear elastic and viscous properties of the components on stress redistribution near inclusions in multicomponent PCMs. As a practical result, one can note the possibility of predicting

the long-term strength of a material when a viscoelastic stress field is known near inclusions. As a conclusion we can notice that the nonlinear viscoelastic model suggested may be useful for prediction of long-term durability and nondestructive control problems of composites.

Концентрація напружень в нелінійних в'язкопружних композитах

Б.П. Маслов

Анотація. *Розв'язано нелінійну в'язкопружну задачу механіки композитів в рамках нелінійної теорії другого порядку. Використано в'язкопружний функціонал для побудови загальних визначальних співвідношень. Розв'язано стохастичну крайову задачу по визначенню концентрації напружень та її релаксації в полімерних композитних матеріалах (ПКМ). Для отримання повної системи рівнянь в'язкопружності другого порядку використано метод послідовної апроксимації. Отримано узагальнення принципу відповідності на нелінійні в'язкоупружні середовища. Визначено усереднені по в'язкопружній матриці і пружним включенням функції релаксації і параметри концентрації напружень. Наведено приклади, що показують важливість взаємного впливу нелінійних пружних і в'язких властивостей складових на перерозподіл напружень поблизу включень в багатокомпонентних ПКМ. В якості практичного результату можна відзначити можливість прогнозування тривалої міцності матеріалу, коли відомо в'язкопружне поле напружень поблизу включень.*

Ключові слова: *в'язкопружний композит; нелінійна деформація; спадкове ядро; ідентифікація; комп'ютерне моделювання*

Концентрация напряжений в нелинейных вязкоупругих композитах

Б.П. Маслов

Аннотация. *Решена нелинейная вязкоупругая задача механики композитов в рамках нелинейной теории второго порядка. Используется вязкоупругий функционал для построения общих определяющих соотношений. Решена стохастическая краевая задача по определению концентрации напряжений и ее релаксации в полимерных композитных материалах (ПКМ). Для вывода полной системы уравнений вязкоупругости второго порядка используется метод последовательной аппроксимации. Получено обобщение принципа соответствия на нелинейные вязкоупругие среды. Определены усредненные по вязкоупругой матрице и упругим включениям функции релаксации и параметры концентрации напряжений. Приведены примеры, показывающие важность взаимного влияния нелинейных упругих и вязких свойств составляющих на перераспределение напряжений вблизи включений в многокомпонентных ПКМ. В качестве практического результата можно отметить возможность прогнозирования долговременной прочности материала, когда известно вязкоупругое поле напряжений вблизи включений.*

Ключевые слова: *вязкоупругий композит; нелинейная деформация; наследственное ядро; идентификация; компьютерное моделирование*

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