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Semi-analytical implicit direct time integration scheme on example of 1-D wave propagation problem

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Abstract. *The most common approach in dynamic analysis of engineering structures and physical phenomenas consists in finite element discretization and mathematical formulation with subsequent application of direct time integration schemes. The space interpolation functions are usually the same as in static analysis. Here on example of 1-D wave propagation problem the original implicit scheme is proposed, which contains the time interval value explicitly in space interpolation function as results of analytical solution of differential equation for considered moment of time. The displacements (solution) at two previous moments of time are approximated as polynomial functions of position and accounted for as particular solutions of the differential equation. The scheme demonstrates the perfect predictable properties as to dispersion and dissipation. The crucial scheme parameter is the time interval – the lesser the interval the more correct results are obtained. Two other parameters of the scheme – space interval and the degree of polynomial approximation have minimal impact on the general behavior of solution and have influence on small zone near the front of the wave.*

Keywords: *1-D wave equation, implicit, direct time integration, suddenly applied force, numerical dissipation.*

1. Introduction

The transient behavior of various engineering structures and wave propagation are usually performed by method of mode superposition or direct time integration [1]. The mode superposition becomes a cumbersome approach in case of nonlinear effects, thus most of investigations and software are based on direct time integration of finite element equations. Regretfully in analysis of wave propagation problems these solutions are often lack of sufficient accuracy, which is exhibited in artificial period elongations (dispersion) and amplitude decays (dissipation) [2]. The reason of it lies in the spatial discretization and elaboration of finite elements as well as time integration peculiarities.

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Furthermore, the common approach for structural dynamic equations in FEM is independent treatment of spatial finite element description, i.e., in dynamical analysis it is supposed that interpolation functions within each element are time independent and are the same as in static [3]. Yet the theoretical solution of 1-D wave propagation has the form $U(x,t)=f_{1,2}(x\pm ct)$, where x – is the spatial variable, t – is a time, and c is a constant of material. The same is about the general solution by natural modes, where the general term can be approximately presented as:

$$U_n(x,t)=\cos(\omega_n x/c)\cdot\cos(\omega_n t)= \\ =\frac{1}{2}\cos\left(\frac{\omega_n}{c}(x+ct)\right)+\frac{1}{2}\cos\left(\frac{\omega_n}{c}(x-ct)\right). \qquad (1)$$

Where n is the mode number, and ω_n is the mode frequency. So, the space and time coordinates are interrelated in exact analytical solutions. This means that within the small space element the interpolation functions in FEM should be time dependent too. Such idea was vaguely expressed in book [3].

In this paper we elaborate this idea on the very popular example of 1-D wave propagation problem. A lot of authors demonstrate the advantages and peculiarities of application of their methods of time integration on example of the elastic long rod subjected to variable force, mention here only some recent works [4–15]. There are, at least, three reasons for that. First, it has a remarkable and simple analytical solution to compare with, which can be adjusted to different boundary and initial conditions, to rod consisting of several materials, etc. Second, for suddenly applied force the solution has an abrupt propagation velocity front, which is hard to obtain in numerical approach, so the quality of numeric calculation can be evidently demonstrated. Third, this type of equation and solution is very important for practical water hammer analysis and 1-D gas dynamic of multibranched piping systems [16, 17].

A few words about the time integration schemes. The equation of equilibrium is considered in consequent discrete points of time. In structural dynamic the order of differential equation with respect to time is equal to 2. It means that minimal number of required consequent time points in any scheme is 3. In general, step-by-step integration methods are divided into two types: explicit and implicit. The explicit methods are considered to be much simpler because they do not require the factorization of the effective stiffness matrix. On other hand they are only unconditionally stable, requires very small steps of time and, what is important in our opinion, they do not eventually lead to the static solution, when the outer load stabilizes in time. Implicit methods are more versatile, although are much expensive.

In our work we will use the simplest implicit central difference scheme [18], so we will not discuss here the abundance of modern developments in time integration schemes [4–15]. The main emphasis of paper is a new technique of accounting for the time integration interval Δt in the space interpolation functions. In this sense the idea of work is similar to spectral element technique [19], where the of frequency is employed in space interpolation functions. The principal difference consists in necessity to account for the states achieved in two previous points of time.

2. Semianalytical method

Start from the general equation of 1-D wave propagation in rod with length *L*, Fig 1. The derivative of displacement, $U(x,t)$ is related with inner force $N(x,t)$, and derivative of force is related with the acceleration, so:

$$\frac{dU(x,t)}{dx} = -\frac{N(x,t)}{EF} \tag{2a}$$

$$\frac{dN(x,t)}{dx} = -\rho F \frac{d^2U(x,t)}{dx} \tag{2b}$$

Where the axis *x*, force N and displacement U are directed from the left to right, Fig 1. Excluding the inner force from these equations we get the 1-D wave propagation equation:

$$\frac{d^2U(x,t)}{dt^2} = c^2 \frac{d^2U(x,t)}{dx}, \tag{2c}$$

$$c^2 = E/\rho . \tag{2d}$$

Where E is the module of elasticity, ρ is the density, F is the rod area.

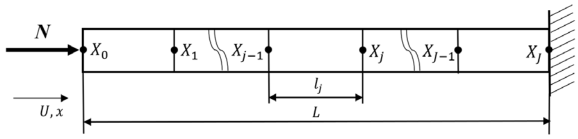

Fig. 1. General scheme of rod

According to simplest central difference scheme consider the looking for function U only in discrete points of time $t_0, t_1, ..., t_{i-1}, t_i, ...$ As in all numerical methods break down the whole length of rod by points $X_0, X_1, ..., X_j, ..., X_J$ on J elementary sections (elements). For each element, j, introduce the local system of coordinate *x*:

$$0 \le x = X - X_{j-1} \le l_j; \qquad x_j - x_{j-1} = l_j \tag{2e}$$

where l_j is the length of element j. Withing each element j the looking for function U in each point of time t_i is considered as a continuous function of x.

For each space element *j* the left side of wave equation (2c) can be presented according to the simplest central difference scheme centered in time point $i-1$, as:

$$\frac{d^2U^i(x,t)}{dt^2}\Big|_{i-1} = \frac{U_i^j(x) - 2U_{i-1}^j(x) + U_{i-2}^j(x)}{\Delta t^2}. \tag{3a}$$

The general scheme of solution of (2c) to be the implicit one, we need to take the right hand of (2c) in the point of time equal to i. So, according to the simplest implicit scheme we can rewrite (2c):

$$\frac{U_i^j(x) - 2U_{i-1}^j(x) + U_{i-2}^j(x)}{\Delta t^2 c^2} = \frac{d^2U_i^j(x,t)}{dx^2}. \tag{3b}$$

In subsequent analysis we mostly omit the upper index j, it will be used only when several length elements are considered. Expression (3b) can be rewritten as:

$$\frac{d^2U_i(x)}{dx^2} - b^2U_i(x) = -b^2Z_{i-1}(x); \quad 0 \le x \le l_j. \tag{3c}$$

Where the constant b is given by:

$$c^2 \cdot \Delta t^2 = b^{-2}. \tag{3d}$$

And auxiliary function $Z(x)$ is determined from two previous moments of time, and is considered to be known:

$$Z_{i-1}(x)=2U_{i-1}(x)-U_{i-2}(x). \tag{3e}$$

Equation (3c) is the main equation of analysis. Its solution is presented as the sum of homogeneous solution, $\bar{U}_i(x)$, and partial solution of nonhomogeneous equation, $\hat{U}_i(x)$:

$$U_i(x)=\bar{U}_i(x)+\hat{U}_i(x). \tag{4a}$$

It will be presented in form suitable for application of transfer matrix method [20, 21]. According to it [21], solution $\bar{U}_i(x)$ is to satisfy the initial conditions (at the beginning of the given element, j):

$$\bar{U}_i(x=0)=U_b;\quad \bar{N}_i(x=0)=N_b. \tag{4b}$$

The respected homogeneous solution in any point x can be expressed through the given initial condition, and is written below:

$$\begin{pmatrix}\bar{U}_i(x)\\ \bar{N}_i(x)\end{pmatrix}=\begin{bmatrix}K_1(x) & -K_2(x)/(EF)\\ -K_1'(x)EF & K_2'(x)\end{bmatrix}\begin{pmatrix}U_{b,i}\\ N_{b,i}\end{pmatrix}. \tag{5a}$$

Where $K_1(x)$ and $K_2(x)$ are so called generalized Krylov's functions, which satisfy to the following initial conditions:

$$K_1(0)=1;\quad K_1'(0)=0;\quad K_2(0)=0;\quad K_2'(0)=1. \tag{5b}$$

And for given differential equation (3c) these functions are:

$$K_1(x)=ch(xb);\ K_2(x)=sh(xb)/b. \tag{5c}$$

These functions have another remarkable property:

$$K_1'(x)=b^2K_2(x);\qquad K_2'(x)=K_1(x). \tag{5d}$$

Next step is to find the partial solution. We look for specific partial solution which is to satisfy the zeroth initial conditions [21]:

$$\hat{U}_i(0)=0,\qquad \hat{N}_i(0)=0. \tag{6a}$$

To find the partial solution of (3c) the function $Z_{i-1}(x)$ should be specified. We require that it should contains the static solution of (2c), and some additional terms. So, present it in simplest form as the polynomial expansions:

$$Z_{i-1}(x)=c_0^{i-1}+c_1^{i-1}x+c_2^{i-1}x^2+c_3^{i-1}x^3+\ldots \tag{6b}$$

Where first two terms $c_0^{i-1}+c_1^{i-1}x$ provides the static solution of (2c). Two other terms are not necessary, and their contribution will be investigated below. Usual partial solution of (3c), $\hat{U}_{us,i}(x)$, with right side taken in form (6b), is following:

$$\hat{U}_{us,i}(x)=c_0^{i-1}+c_1^{i-1}x+c_2^{i-1}\left(x^2+\frac{2}{b^2}\right)+c_3^{i-1}\left(x^3+\frac{6}{b^2}x\right), \tag{6c}$$

$$\frac{\hat{N}_{us,i}(x)}{EF}=-c_1^{i-1}-c_2^{i-1}2x-c_3^{i-1}\left(3x^2+\frac{6}{b^2}\right). \tag{6d}$$

In space point $x=0$ it gives nonzero values:

$$\hat{U}_{us,i}(0)=c_0^{i-1}+\frac{2}{b^2}c_2^{i-1};$$
$$\hat{N}_{us,i}(0)=-EF\left(c_1^{i-1}+c_3^{i-1}\frac{6}{b^2}\right). \tag{6e}$$

To satisfy zeroth initial condition it should be supplemented by general homogenous solution with unknown coefficients A and B:

$$\hat{U}_{us,i}(x)=c_0^{i-1}+c_1^{i-1}x+c_2^{i-1}\left(x^2+\frac{2}{b^2}\right)+$$
$$+c_3^{i-1}\left(x^3+\frac{6}{b^2}x\right)+AK_1(x)+BK_2(x). \tag{6f}$$

Accounting for the properties of the generalized Krylov's functions and initial values of usual partial solution (6d) and (6e), we get:

$$\hat{U}_i(x)=c_0^{i-1}(1-K_1(x))+c_1^{i-1}(x-K_2(x))+$$
$$+c_2^{i-1}\left(x^2+\frac{2(1-K_1(x))}{b^2}\right)+c_3^{i-1}\left(x^3+\frac{6(x-K_2(x))}{b^2}\right). \tag{6g}$$

$$\frac{\hat{N}_i(x)}{EF}=c_0^{i-1}K_1'(x)-c_1^{i-1}(1-K_2'(x))-$$
$$-c_2^{i-1}\left(2x-\frac{2K_1'(x)}{b^2}\right)-c_3^{i-1}\left(3x^2+\frac{6(1-K_2'(x))}{b^2}\right). \tag{6h}$$

So, expressions (5a) with (6g) and (6h) give the complete solution for given section (element) according to (4a).

Now consider how to get the solution for the whole rod at the given point of time. Thus, the lower indexes i will be omitted below and upper indexes j will be retained. Rewrite the complete solution for the element j in the general form:

$$\begin{pmatrix}U^j(x)\\ N^j(x)\end{pmatrix}=\begin{bmatrix}K_1(x) & -K_2(x)/(EF)\\ -K_1'(x)EF & K_2'(x)\end{bmatrix}\begin{pmatrix}U_b^j\\ N_b^j\end{pmatrix}+\begin{pmatrix}\hat{U}^j(x)\\ \hat{N}^j(x)\end{pmatrix}. \tag{7a}$$

This solution allows to relate the vector of state in any point x of the section j with the state at initial point $x=0$. So, it can establish the relation for the last point $x=l_j$, too:

$$\begin{pmatrix}U_e^i\\ N_e^i\end{pmatrix}=\begin{pmatrix}U^j(l_j)\\ \bar{N}_i(l_j)\end{pmatrix}=$$
$$=\begin{bmatrix}K_1(l_j) & -K_2(l_j)/(EF)\\ -K_1'(l_j)EF & K_2'(l_j)\end{bmatrix}\begin{pmatrix}U_b^j\\ N_b^j\end{pmatrix}+\begin{pmatrix}\hat{U}^i(x)\\ \hat{N}^i(x)\end{pmatrix} \tag{7b}$$

Now consider the border between the elements j and $j+1$. The solution should be continuous, so we should write the conditions of continuity of the vector of state, which is characterized by displacement and force:

$$\begin{pmatrix} U_b^{j+1} \\ N_b^{j+1} \end{pmatrix} = \begin{pmatrix} U_e^{j} \\ N_e^{j} \end{pmatrix}. \quad (7c)$$

The above values of displacement and force at the ends of each section are considered as unknowns. To form the systems of equations for them we use equations (7b) and (7c), supplemented by two boundary conditions for the whole rod – one on the left side and one at the right side of the rod. For example, for considered below free-clamped rod these conditions are:

$$N_b^1 = N_L; \quad U_e^J = 0. \quad (7d)$$

The only remained issue to be solved is a polynomial presentation of the function $Z_{i-1}(x)$ in form (6b), given that it is determined from (3e), which terms were derived in form (7a).

Rewrite (7a) in other form:

$$U_i(x) = \alpha_i K_1(x) + \beta_i K_2(x) + \hat{U}_{us,i}(x); \quad (8a)$$

$$N_i(x) = -\alpha_i EFb^2 K_2(x) - \beta_i EFK_1(x) + \hat{N}_{us,i}(x). \quad (8b)$$

Where

$$\alpha_i = U_{0,i} - c_0^{i-1} - \frac{2}{b^2} c_2^{i-1}; \beta_i = -\frac{N_{0,i}}{EF} - c_1^{i-1} - c_3^{i-1}\frac{6}{b^2}. \quad (8c)$$

Expand functions $K_1(x)$ and $K_2(x)$ in polynomial series of third degree. The simplest way is to use well-known Tailor's expansion. The best way is to get the integrally weighted expressions. So, present $K_1(x)$ in form:

$$ch(xb) = K_1(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3. \quad (9a)$$

Multiply both sides of (9a) by 1 and integrates over x from $x=1$ to $x=l$, we get:

$$\frac{sh(lb)}{b} = f_0 l + f_1\frac{l^2}{2} + f_2\frac{l^3}{3} + f_3\frac{l^4}{4}. \quad (9b)$$

Then multiply by x and similarly integrate:

$$\frac{lsh(lb)}{b} - \frac{ch(lb)-1}{b^2} = f_0\frac{l^2}{2} + f_1\frac{l^3}{3} + f_2\frac{l^4}{4} + f_3\frac{l^5}{5}. \quad (9c)$$

Next multiplying by x^2 and integrating, we get:

$$\frac{l^2 sh(lb)}{b} - \frac{2lch(lb)}{b^2} + \frac{2sh(lb)}{b^3} = f_0\frac{l^3}{3} + f_1\frac{l^4}{4} + f_2\frac{l^5}{5} + f_3\frac{l^6}{6}. \quad (9d)$$

Similarly, for x^3:

$$\frac{l^3 sh(lb)}{b} - \frac{3l^2 ch(lb)}{b^2} + \frac{6lsh(lb)}{b^3} - \frac{6(ch(lb)-1)}{b^4} = $$
$$= f_0\frac{l^4}{4} + f_1\frac{l^5}{5} + f_2\frac{l^6}{6} + f_3\frac{l^7}{7}. \quad (9e)$$

Four coefficients of expansion of $K_1(x)$ are derived from these four equations (9b–e).

In similar way present $K_2(x)$ as a polynomial expansion:

$$K_2(x) = \frac{sh(xb)}{b} = h_0 + h_1 x + h_2 x^2 + h_3 x^3. \quad (10a)$$

Where the coefficients h_k are derived from the following four equations:

$$\frac{ch(lb)}{b^2} - \frac{1}{b^2} = h_0 l + h_1\frac{l^2}{2} + h_2\frac{l^3}{3} + h_3\frac{l^4}{4}; \quad (10b)$$

$$\frac{lch(lb)}{b^2} - \frac{sh(lb)}{b^3} = h_0\frac{l^2}{2} + h_1\frac{l^3}{3} + h_2\frac{l^4}{4} + h_3\frac{l^5}{5}; \quad (10c)$$

$$\frac{l^2 ch(lb)}{b^2} - \frac{2lsh(lb)}{b^3} + \frac{2(ch(lb)-1)}{b^4} = $$
$$= h_0\frac{l^3}{3} + h_1\frac{l^4}{4} + h_2\frac{l^5}{5} + h_3\frac{l^6}{6}; \quad (10d)$$

$$\frac{l^3 ch(lb)}{b^2} - \frac{3l^2 sh(lb)}{b^3} + \frac{6lch(lb)}{b^4} - \frac{6sh(lb)}{b^5} = $$
$$= h_0\frac{l^4}{4} + h_1\frac{l^5}{5} + h_2\frac{l^6}{6} + h_3\frac{l^7}{7}. \quad (10e)$$

Now present the calculated function $U_i(x)$ on *i*-time step at each space section as a polynomial series:

$$U_i(x) = u_0^i + u_1^i x + u_2^i x^2 + u_3^i x^3. \quad (11a)$$

Comparing it with (8a) and accounting for (6c) we can get the approximation for each coefficient u_k^i:

$$u_0^i = \alpha_i f_0 + \beta_i h_0 + c_0^{i-1} + c_2^{i-1}\frac{2}{b^2}; \quad (11b)$$

$$u_1^i = \alpha_i f_1 + \beta_i h_1 + c_1^{i-1} + c_3^{i-1}\frac{6}{b^2}; \quad (11c)$$

$$u_2^i = \alpha_i f_2 + \beta_i h_2 + c_2^{i-1}; \quad (11d)$$

$$u_3^i = \alpha_i f_3 + \beta_i h_3 + c_3^{i-1}. \quad (11e)$$

The next step is compilation of coefficients c_k^i, $k = 0,1,2,3$ based on results of calculations at time step $i-1$, and i. According to definition of $Z_{i-1}(x)$ by (3e), we can write:

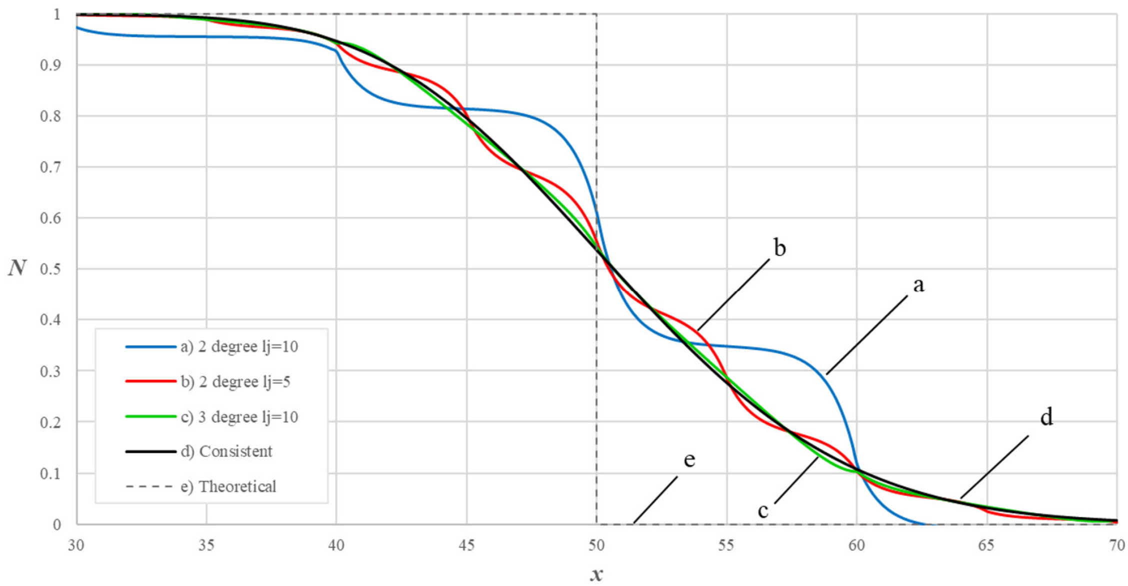

Fig. 2. Dependance of axial force from axial coordinate at the time equal to 50s with a time step $\Delta t = 1$: *a*) case 6, $l = 10$, $K = 2$; *b*) case 4, $l = 5$, $K = 2$; *c*) case 7, $l = 10$, $K = 3$; *d*) all other considered cases; *e*) theoretical solutions

$$c_k^i = 2u_k^i - u_k^{i-1}. \tag{11f}$$

Thus, all steps in formulation of semi analytical approach are described. It operates by the following parameters: 1) time step, Δt; 2) length of element l_j; 3) number of terms of expansions of previous state (6b) – two, three or four. In the following we will explore their influence on the accuracy of solution.

3. Examples for suddenly applied force

For simplicity, in all examples we consider the clamped-free bar with a clamped right side, and left side is loaded by some transient force $P(t)$, Fig 1. All constants related to material and section (E, ρ, F) are taken to be 1, so the constant c is also equal to 1. The length of rod, L, is equal to 100.

In all following examples we will compare our solution with Fourier based mode superposition solution. Thus, give some preliminary results and constants for this task. The natural forms, $\Phi_n(x)$, are given the by following expression:

$$\Phi_n(x) = \cos\cos\frac{\omega_n x}{c} = \cos\cos(\omega_n x). \tag{12a}$$

Where the frequencies ω_n are calculated from:

$$\omega_n = \frac{c}{L}\frac{(2n+1)\pi}{2} = \frac{(2n+1)\pi}{200}, \quad n = 1, 2, 3... \tag{12b}$$

3.1 The force is studently applied at the moment $t = 0$. The analytical mode superposition solution is given by the following expression:

$$U(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{L\omega_n^2}(1 - \cos\omega_n t)\cdot\Phi_n(x)\right). \tag{13}$$

Our goal is to investigate the influence of the semi analytical scheme parameters on accuracy of calculation. There are three main groups of them: time step, Δt; length of element, l_j; and power, K, of previous history expansion – $K = 2$ terms (static like), $K = 3$ terms, or $K = 4$ terms. As we see later, the most important is the time step. Nevertheless, start our investigation from two last groups.

3.1.1. So, fix the time step, $\Delta t = 1$, and consider several different combinations of constant lengths of element, $l_j = l$ and the degrees of expansion. Present results of force determination, $N(x, t = 50)$ for moment of time $t = 50s$ in the vicinity of point $X = 50m$.

Several cases are considered, and results for them are shown on Fig. 2. These cases are: 1) $l = 0.1$ (1000 elements), $K = 2$; 2) $l = 1$ (100 elements), $K = 2$; 3) $l = 2$ (50 elements), $K = 2$; 4) $l = 5$ (20 elements), $K = 2$; 5) $l = 5$, $K = 3$; 6) $l = 10$ (10 elements), $K = 2$; 7) $l = 10$, $K = 3$; 8) $l = 10$, $K = 4$. At this stage of analysis, we are only interested in the numerical consistence, or by other words coincidence of results for different parameters of numerical scheme. Draw a few conclusions from these results.

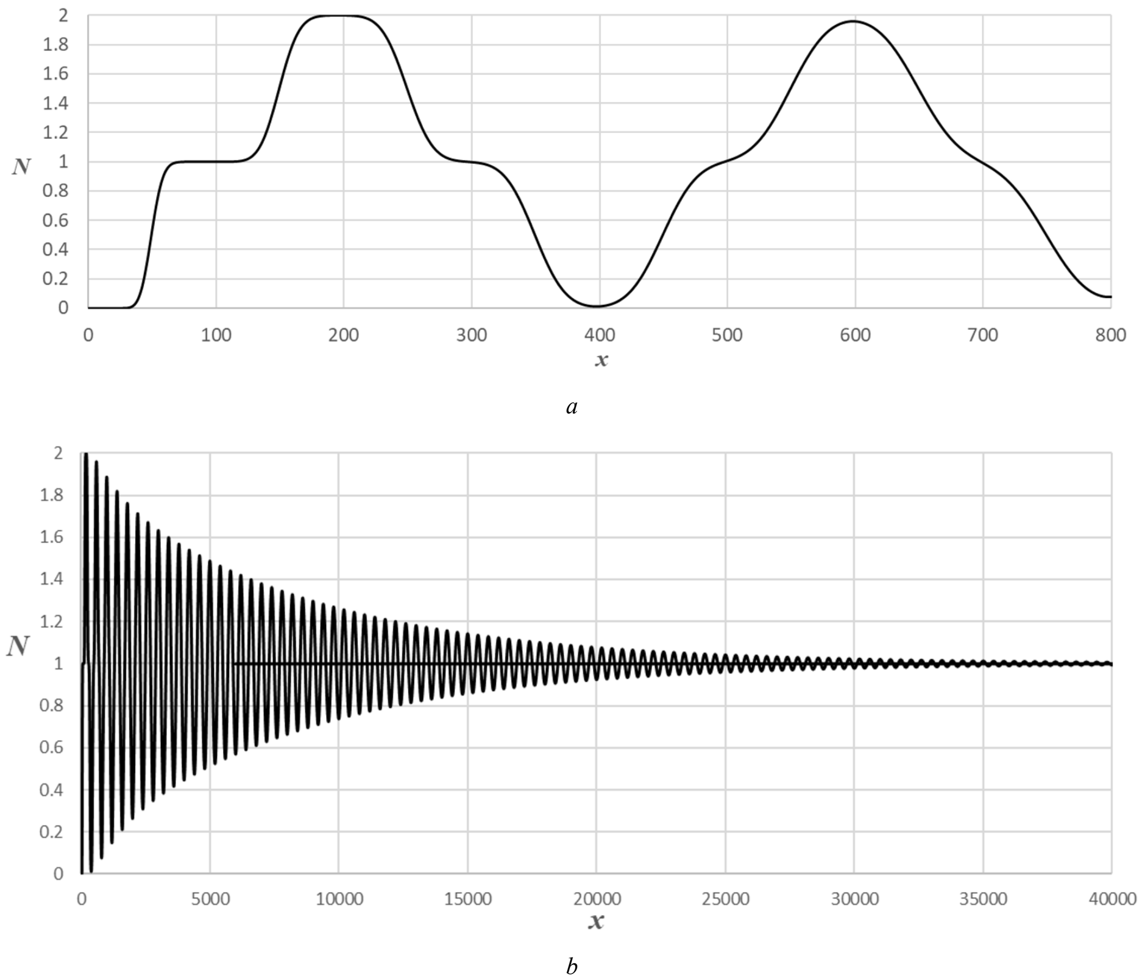

Fig. 3. Axial force in point $X = 50m$ for case $l = 1$, $\Delta t = 1$, $K = 4$: *a*) $0 \le t \le 600$, *b*) $0 \le t \le 40000$

Two terms (static like) expansion of previous history provide the consistent results if the length of element is smaller than 2. More generally, 2 terms consistency condition is:

$$0 < lb \le 2 \quad \text{or} \quad 0 < l \le 2 \cdot \Delta t \cdot c\,. \tag{14a}$$

Similarly, three terms expansion leads to consistency at slightly larger length of element (lesser number of elements):

$$0 < l \le 5 \cdot \Delta t \cdot c\,. \tag{14b}$$

The increase of number of terms allows to further increase the length of element. Thus for 4 terms we get the following condition:

$$0 < l \le 10 \cdot \Delta t \cdot c\,. \tag{14c}$$

Actually, the above conditions (14) justify the idea (name) of the proposed numerical scheme – semi-analytic one with respect to space coordinate x. It means that meshing in space practically has no influence on results. Also, conditions (14) will be used in analysis of influence of time step, Δt.

Interesting to note, that proposed scheme gives no complications during application of elements of different length. The only restriction is that each element should satisfy the condition of consistency.

3.1.2. Analysis of spurious oscillations and amplitude decay (dissipation). They usually occur in most numerical schemes. So, consider them on example of case $l = 1$, $\Delta t = 1$, $K = 2$. Here we fix the point of space $X = 50m$, and build the graph of axial force in it with time, Fig. 3. Fig. 3, *a* show the time dependance of force in time range $0 \le t \le 800$. As we see, there are no spurious oscillations, which is the remarkable property of the proposed scheme. Fig. 3, *b* shows another remarkable property – the results, while decaying, still tend to the static solution. This means that proposed scheme is consistent with static solution.

3.1.3 Analysis of dispersion. To characterize it quantitively let us formulate some subjective criteria. We let that full dispersion does occur, if the maximum of force become smaller than 2. Introduce the time period, T_0 :

$$T_0 = 2L/c \tag{15a}$$

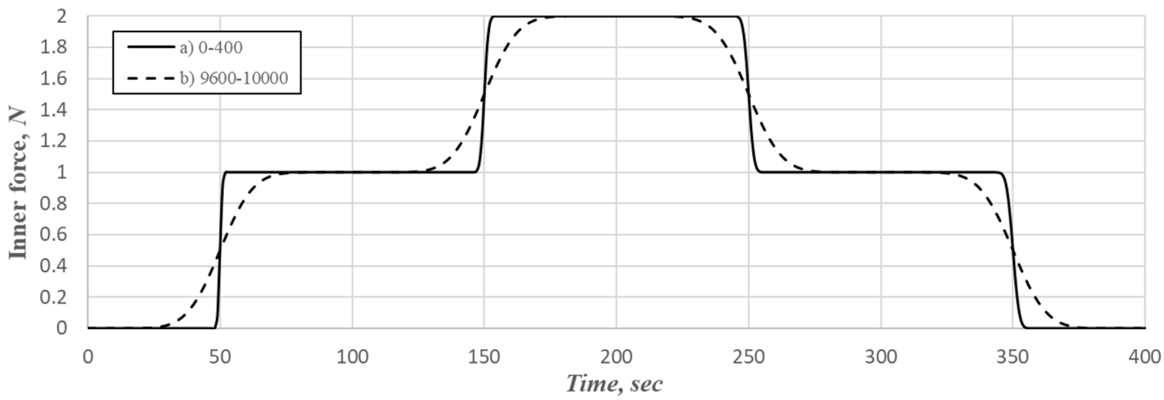

Fig. 4. Axial force in point $X = 50m$ for case $\Delta t = 0.02$, $l = 0.2$, $K = 4$ for two time windows: *a*) $0 \le t \le 400$, *b*) $0 \le t - 47T_0 \le 400$

which is required for the wave to go and return along the entire rod. Here $T_0 = 200s$. So, the question is how many time periods the wave can travel along the rod until the dispersion would occur in dependence with chosen time step? The answers are given in Table 1, which we obtained by calculation of the critical time T_c at which the full dispersion occurs.

Table 1. Number of periods until the dispersion would occur

$\Delta t, s$	Time of full dispersion, T_c, s	Number of T_0 at full dispersion
1	400	2
0.5	800	4
0.25	1600	8
0.2	2000	10
0.1	4000	20
0.05	8000	40
0.02	20000	100
0.01	40000	200

As we see, the number of time periods at full dispersion is inversely proportional to Δt. Let us justify this empirical conclusion by simplified considerations. As we know the implicit central difference scheme possesses the accuracy, ε, (error at one time step realization) proportional to Δt^2, i.e.:

$$\varepsilon = \gamma \Delta t^2 . \tag{15b}$$

Where γ is coefficient of proportionality. So, the critical error E can be attained at time step number N_c:

$$N_c \varepsilon = E . \tag{15c}$$

From the other hand, the critical time, T_c, is proportional to the product of time step and number of steps:

$$T_c = N_c \cdot \Delta t = \frac{E}{\gamma \Delta t} . \tag{15d}$$

This simplifying consideration justifies the numerically observed behavior.

3.1.4. Detailed behavior at $\Delta t = 0.02$, $l = 0.2$ (500 elements), $K = 4$. This example is chosen to illustrate the ability of the scheme to reflect the ideal theoretical behavior of the wave front. Fig. 4 shows the axial force in point $X = 50m$ at two times windows. First one embraces the period $0 \le t \le 400$ and second one - $0 \le t - 47T_0 \le 400$. In this sense the semianalytical scheme resemble the Fourie analysis – the more terms are used the more accurate solution can be derived.

3.2. Consider force impulse with restricted duration. The boundary initial conditions are given below:

$$U_R(100, t) = 0; \tag{16a}$$

$$\frac{dU_L}{dx}(0,t) = N_L(0,t) = \begin{cases} -\dfrac{1}{ES} = -1, & t \le 40 \\ 0, & t > 40 \end{cases} \tag{16b}$$

The similar examples are very rarely analyzed in literature. It is interesting that here we have both the fore and rear fronts of the wave.

3.2.1 The analytical mode superposition solution for initial 40 seconds is given by expression (13). As to the subsequent behavior we introduce the auxiliary time τ:

$$t = \tau + 40 . \tag{17a}$$

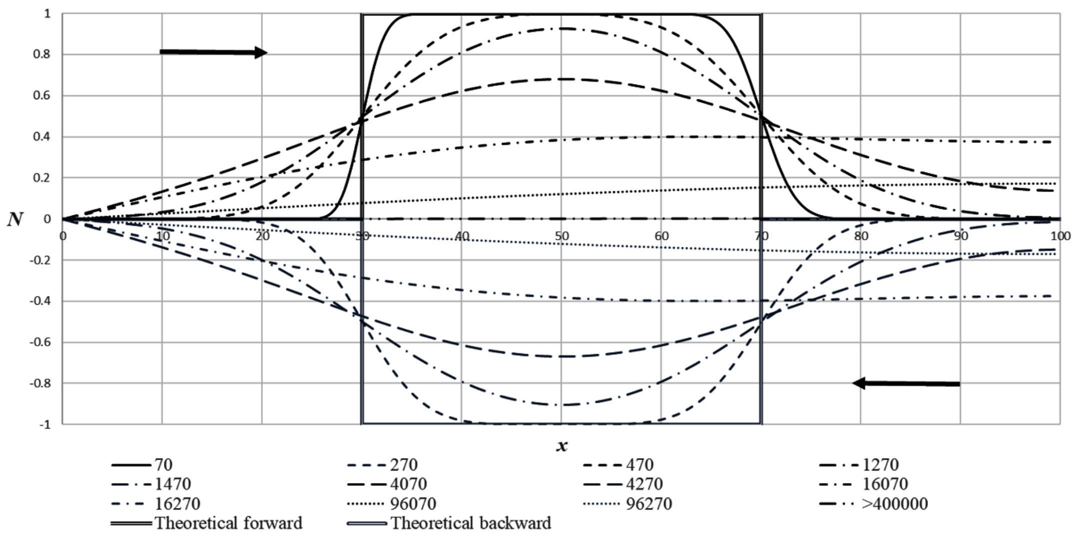

Fig. 5. Forward and backward waves at short impulse loading shown for various moments of time

Then the following solutions as to displacement and force can be derived:

$$U(x,\tau)=\sum_{n=1}^{\infty}\frac{2}{L\omega_n^2}\big((1-\cos\omega_n 40)\cos\omega_n\tau+ +\sin\omega_n 40\sin\omega_n\tau\big)\cdot\Phi_n(x)\,. \tag{17b}$$

$$N(x,\tau)=\sum_{n=1}^{\infty}\frac{2}{Lc\omega_n^2}\big((1-\cos\omega_n 40)\cos\omega_n\tau+ +\sin\omega_n 40\sin\omega_n\tau\big)\cdot\sin\sin\frac{\omega_n x}{c}\,. \tag{17c}$$

3.2.2. Numerical solution. Take a relatively coarse mesh, namely $\Delta t=0.1$, $l=0.5$ (200 elements, $b=5$), $K=4$. The results of calculation of forces $N(x)$ at fixed moments of time are shown on Fig. 5. As it shown here the fore and rear fronts are determined with similar accuracy. The results are shown for the moments of time equal to:

$$t_{forw}=70+2kT_0 \tag{18a}$$

$$t_{back}=70+(2k+1)T_0 \tag{18b}$$

Where t_{forw} are the moments of time when the wave goes forward (from left to right), and t_{back} are the moments when reflected wave goes backward. As in above example, the results are quite logical, they show the similar tendency of decay behavior, and eventually they tend to zero – static solution for this case. No spurious oscillations are noticed at any space and time points.

4. Conclusion

Here the original implicit direct time integration scheme is proposed. Contrary to existing approaches it initially applies the central difference scheme to the second time derivative, and then consider the static like (time independent) boundary problem, where the time interval is the constant parameter in space differential equation. So, the space interpolation functions become dependent from the time interval. The solutions at two previous moments of time are approximated by polynomials of K degree and accounted for as the particular solutions of the differential equation.

The numerical example for suddenly applied force shows a good consistency of approach which demonstrate no spurious oscillations. The main parameter which predetermines the accuracy and decay behavior is the time step – the decay is inversely proportional to it. As to space interval and degree of approximation polynomial – they have auxiliary significance and should satisfy some minimum requirements only. For example, for $K=4$ space interval should satisfy condition (14c); for $K=3$ – condition (14b); and for $K=2$ – condition (14a).

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Напіваналітична неявна схема прямого інтегрування по часу на прикладі одновимірної задачі поширення хвилі

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Анотація. *Найпоширеніший підхід у динамічному аналізі інженерних конструкцій і фізичних явищ полягає в кінцево-елементній дискретизації та математичному формулюванні з подальшим застосуванням схем прямого інтегрування по часу. Функції просторової інтерполяції зазвичай такі ж, як і в статичному аналізі. Тут на прикладі одновимірної задачі про поширення хвилі запропоновано оригінальну неявну схему, яка містить значення часового інтервалу явно в просторовій інтерполяційній функції як результат аналітичного розв'язку диференціального рівняння для розглянутого моменту часу. Переміщення (розв'язок) у два попередні моменти часу апроксимуються як поліноміальні функції положення та враховуються як часткові розв'язки диференціального рівняння. Схема демонструє ідеальні передбачувані властивості щодо дисперсії та дисипації. Вирішальним параметром схеми є часовий інтервал – чим менше інтервал, тим точніші результати. Два інших параметри схеми – просторовий інтервал і ступінь поліноміальної апроксимації мають мінімальний вплив на загальну поведінку розв'язку і впливають на малу зону біля фронту хвилі.*

Ключові слова: *одновимірне хвильове рівняння, неявне, пряме інтегрування в часі, раптово прикладена сила, чисельна дисипація.*