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Applications of randomly selected sets of exact Voight's solutions for vibration of thin plates

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Abstract. *The principally new method of selected exact solutions, SES, for plate vibration based on fundamental solutions of Voigt is suggested. In contrast to similar known methods, it employs the frequency dependent functions for both space coordinates. The sets of exact solutions which depends on some arbitrary chosen parameters are constructed. This allows to choose any number of exact solutions, while the required number of them depends on the boundary conditions which should satisfy in considered collocation points. The efficiency of method is demonstrated for the most unfavorable case of all sides clamped rectangular plate. Nevertheless, the accuracy is quite satisfactory for first six natural frequencies even for relatively small number of collocation boundary points, and testify about big prospects as to application for complex structures, different geometries, various boundary conditions.*
Additionally two variants of the Galerkin method are realized and compared. First one, employs the exponential functions, while the second one –the very popular beam functions. The calculation results show the superiority of first variant as in technical realization as in accuracy, and in further applications in structural mechanics.

Keywords: *rectangular plate, free vibrations, clamped-clamped plate, Galerkin method, Voigt solution.*

Introduction

In spite of very long history of investigation the vibration of plates is still a topic of primary interest important for various engineering application, for example, in solar plates, circuit plates, robots, aircraft, etc. Sometimes these plates are constrained by others structural elements, which should be considered and modeled withing one comprehensive calculational scheme [1]. Such necessity restricts the efficiency of very popular numerical techniques (finite difference, finite element methods) [1], and promote the development of analytical and semianalytical methods. Except for the huge practical significance, this topic is the most efficient educational example for students for the study of partial differential equation eigenvalue problems [2]. Whatever the sophisticated and versatile the method might be for various geometrical and materials properties of structure, its peculiarities and efficiency are usually studied on example of rectangular isotropic plate.

1. The most popular semianalytical methods, in our opinion, are Ritz methods, RM, and weighted residual methods, WRM. RM is based on minimization of functional of energy over the whole area of plate. WRM minimizes the governing differential equation (residual) of the theory of plate, and this is achieved by multiplication by special weight functions [3] and integration over the area. Both methods employ so-called trial functions, which should to satisfy the required boundary conditions. They should form the systems of independent and, preferably, orthogonal functions [3]. These two approaches have a lot of common features in their realization, complexity and application [4], yet RM gives the assessment for eigenvalues from above, while WRM approaches the correct values from below.

The trial functions predetermine the peculiarities and essential difference between various methods. The comprehensive theoretical analysis and investigation of influence of their choice on the computational efficiency on example of clamped rectangular plate was undertaken in work [5]. Six different trial functions were investigated in Ritz method: 1) characteristic functions [6], 2) modified characteristic functions [7], 3) orthogonal polynomials [8],

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4) nonorthogonal polynomials [9], 5) product of trigonometric functions [10], and 6) static beam functions [11]. Regretfully the problems of convergence with respect to the number of trial functions were mostly analyzed there, and modified characteristic functions were stated to exhibit the best results. As to characterized functions, they were failed at very low number of applied functions (it was stated [5] that already 9 terms with respect to each variable lead to appearance of complex eigenvalues).

Nevertheless, the characteristic functions or, by other words, normalized eigenfunctions exactly satisfying the equation of motion of a freely vibrating, uniform beam [12] (beam functions) are very popular in literature even for other problems in structural dynamics, for example, for the cylindrical shell problem [13]. Their application has started from the work of Young [6] and subsequently were efficiently applied in classical work of Leissa [12], where the practical results for rectangular plates at different boundary condition were obtained. Their application requires the preliminary finding of roots for transcendental equations of beam vibration, and analytical integration of different beam functions [14].

From other hand, the principally new family of trial functions were suggested by present authors in works [15–17] where the specially constructed exponential functions, EF, were successfully applied in Galerkin method for various structural mechanics problems. Remind that Galerkin method is a special kind of WRM, where the weight functions coincide with trial ones [3]. So, it is interesting here to compare the results of application of above two kinds of trial functions (EF and beam functions) in Galerkin method. This predetermined the first objective of paper.

2. The notion of analytical solutions for thin plate vibration equation we attribute to original work of Voigt [18] written as far as in 1893. Considering that solution is a product of two exponential (real or complex) functions of separate variables x and y he obtained the simple relations between coefficients of these exponents with natural frequency. For example, for each fixed coefficient in x – dependent exponential function four related y – dependent functions were derived. In work of Leissa [12] this technique was applied for case of plate with two opposite simple supported edges. This allowed to choose the x-functions as family of sinus functions and obtain the respected 4-terms y– dependent complementary solution, which is able to satisfy any arbitrary 4 boundary conditions on two other opposite sides. Actually, the idea of Leissa is a generalization of the static approach of Levi [19]. Later on, it was shown by Bert and Malik [20] that similar exact solutions can be constructed for cases when two opposite sides are either simple supported or guided (where the slope and the shear force are zero).

Gorman presented the concept of the method of superposition and its potential applications in obtaining the accurate analytical solutions for rectangular thin plates with arbitrary combinations of classical boundary conditions [21]. He considered that general solution consists of 4 parts (building blocks) and dealt with each separately: symmetric mode for two coordinates, antisymmetric mode, and two symmetric-antisymmetric mode of deformation.

Gorman methods of superposition found its further development in spectral dynamic stiffness matrix method proposed by Banerjee and co-workers [22]. They analytically rearranged the dependencies between different physical parameters of plate, and made it more suitable for plates of various form. Other notable applications of Voigt solution and Gorman idea of superposition are modern works of Yu and Yin [23], where only two building blocks are constructed based on half-range Fourier cosine series; and exact frequency-domain spectral element model by Kim and Lee [24] which used the fast Fourier transform technique.

In all above analytical methods one function of any coordinate is taken as dependent from the length parameter while another one – is found from the Voigt solution and consequently is dependent from unknown in advance the value of frequency.

The main idea of the present work and its second objective is elaboration of the method of selected exact (Voigt) solution, SES, with combination with the boundary collocations. Both of two product functions are taken as trigonometrical or exponential functions of coordinate multiplied on fraction of unknown frequency. This fraction is taken to be arbitrary in advance for one function, so another function is directly obtained from the first one. So, both functions are frequency dependent.

The accuracy of proposed method is verified on example of clamped rectangular plate by comparison with results of two variants of Galerkin method. Note, that chosen geometry is very popular in practical verifications of various methods, because it usually gives the maximal discrepancies between results [5, 25].

Method of selected exact solutions, SES

The governing differential equation for thin-walled plate is obtained by considering that the function of transverse deflection $W(x,y,t)$ is proportional to $\sin(\varphi t)$:

$$\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2 \partial y^2}+\frac{\partial^4 W}{\partial y^4}=\omega^2 W=\Omega^4 W, \tag{1a}$$

where φ is the frequency, and for convenience the dimensionless, ω, and conventional frequencies, Ω are introduced, which are interrelated as:

$$\omega^2=\frac{\rho h}{D}\varphi^2=\Omega^4 \tag{1b}$$

and $D=\dfrac{Eh^3}{12(1-\nu^2)}$ is the bending rigidity ρ, E, ν , are materials characteristics: density, module of elasticity, Poisson's coefficient, h is wall thickness. Besides, deflections $W(x,y)$ should satisfy the boundary conditions.

The proposed method is based on general solution of Voigt. Assume that:

$$W(x,y)=X(x)Y(y). \tag{2a}$$

If we take that:

$$X(x)=e^{\lambda_x x}, \quad Y(y)=e^{\lambda_y y}, \tag{2b}$$

then, inserting (2a) and (2b) into (1a), the following relations between the parameters of solutions is obtained:

$$\lambda_x^2+\lambda_y^2=\pm\Omega^2. \tag{2c}$$

In particular, for case of simply supported opposite sides of plate along the lines $x=0$, and $x=a$, take, as in work [12], that $X(x)=\sin(\lambda_x x), \lambda_x=na/\pi$, where n is integer, then:

$$W(x,y)=\sin(\lambda_x x)\Big[A_1\sin\Big(\sqrt{\Omega^2-\lambda_x^2}\,y\Big)+ +A_2\cos\Big(\sqrt{\Omega^2-\lambda_x^2}\,y\Big)+A_3 ch\Big(\sqrt{\Omega^2+\lambda_x^2}\,y\Big)+A_4 sh\Big(\sqrt{\Omega^2+\lambda_x^2}\,y\Big)\Big]. \tag{2d}$$

Then four unknowns $A_i\ (i=\overline{1,4})$ are determined from boundary conditions on two other opposite sides. In contrast to the conventional methods [12, 20–24] we present both functions of coordinates as:

$$X(x)=e^{\alpha\Omega x}, \quad Y(y)=e^{\beta\Omega y}, \tag{3a}$$

$$\alpha^2+\beta^2=\pm1. \tag{3b}$$

Thus, both of two parameters α and β are related to the looking for frequency Ω, rather than to the plate dimensions. The idea of method of SES lies in that the value of α is fixed and taken as:

$$\alpha=\pm\gamma, \quad \alpha=\pm i\gamma, \tag{4a}$$

where $0<\gamma<\sqrt{0.5}$, $i=\pm\sqrt{-1}$. Then, according to (3b), the value of β can be determined by two dimensionless parameters:

$$\hat{\gamma}=\sqrt{1-\gamma^2}; \quad \tilde{\gamma}=\sqrt{1+\gamma^2}. \tag{4b}$$

Form and numerate the following solutions of equations (1a). At beginning write the solutions which depends from γ and $\hat{\gamma}$. We have set of 8 solutions $f_i(\Omega,x,y,\gamma,\hat{\gamma})$, where Ω is usually omitted from the designation of the functions:

$$\left[\begin{array}{l} f_1=\sin(\gamma\Omega x)\cdot\sin(\hat{\gamma}\Omega y);\ f_2=\sin(\gamma\Omega x)\cdot\cos(\hat{\gamma}\Omega y);\\ f_3=\cos(\gamma\Omega x)\cdot\sin(\hat{\gamma}\Omega y);\ f_4=\cos(\gamma\Omega x)\cdot\cos(\hat{\gamma}\Omega y);\\ f_5=sh(\gamma\Omega x)\cdot sh(\hat{\gamma}\Omega y);\ f_6=sh(\gamma\Omega x)\cdot ch(\hat{\gamma}\Omega y);\\ f_7=ch(\gamma\Omega x)\cdot sh(\hat{\gamma}\Omega y);\ f_8=ch(\gamma\Omega x)\cdot ch(\hat{\gamma}\Omega y). \end{array}\right. \tag{5a}$$

To provide the equality between the two independent variables x and y, introduce the supplemental eight solutions $h_i(\Omega,x,y,\gamma,\hat{\gamma})$, where x and y are swapped.

$$\left[\begin{array}{l} h_1=\sin(\gamma\Omega y)\cdot\sin(\hat{\gamma}\Omega x);\ h_2=\sin(\gamma\Omega y)\cdot\cos(\hat{\gamma}\Omega x);\\ h_3=\cos(\gamma\Omega y)\cdot\sin(\hat{\gamma}\Omega x);\ h_4=\cos(\gamma\Omega y)\cdot\cos(\hat{\gamma}\Omega x);\\ h_5=sh(\gamma\Omega y)\cdot sh(\hat{\gamma}\Omega x);\ h_6=sh(\gamma\Omega y)\cdot ch(\hat{\gamma}\Omega x);\\ h_7=ch(\gamma\Omega y)\cdot sh(\hat{\gamma}\Omega x);\ h_8=ch(\gamma\Omega y)\cdot ch(\hat{\gamma}\Omega x). \end{array}\right. \tag{5b}$$

Now construct two additional sets of eight solutions $f_i(\Omega,x,y,\gamma,\tilde{\gamma})$ and $h_i(\Omega,x,y,\gamma,\tilde{\gamma})$, which depend on parameters γ and $\tilde{\gamma}$:

$$\left[\begin{array}{l} f_9=\sin(\gamma\Omega x)\cdot sh(\tilde{\gamma}\Omega y);\ f_{10}=\sin(\gamma\Omega x)\cdot ch(\tilde{\gamma}\Omega y);\\ f_{11}=\cos(\gamma\Omega x)\cdot sh(\tilde{\gamma}\Omega y);\ f_{12}=\cos(\gamma\Omega x)\cdot ch(\tilde{\gamma}\Omega y);\\ f_{13}=sh(\gamma\Omega x)\cdot\sin(\tilde{\gamma}\Omega y);\ f_{14}=sh(\gamma\Omega x)\cdot\cos(\tilde{\gamma}\Omega y);\\ f_{15}=ch(\gamma\Omega x)\cdot\sin(\tilde{\gamma}\Omega y);\ f_{16}=ch(\gamma\Omega x)\cdot\cos(\tilde{\gamma}\Omega y). \end{array}\right. \tag{5c}$$

and

$$\left[\begin{array}{l} h_9=\sin(\gamma\Omega y)\cdot sh(\tilde{\gamma}\Omega x);\ h_{10}=\sin(\gamma\Omega y)\cdot ch(\tilde{\gamma}\Omega x);\\ h_{11}=\cos(\gamma\Omega y)\cdot sh(\tilde{\gamma}\Omega x);\ h_{12}=\cos(\gamma\Omega y)\cdot ch(\tilde{\gamma}\Omega x);\\ h_{13}=sh(\gamma\Omega y)\cdot\sin(\tilde{\gamma}\Omega x);\ h_{14}=sh(\gamma\Omega y)\cdot\cos(\tilde{\gamma}\Omega x);\\ h_{15}=ch(\gamma\Omega y)\cdot\sin(\tilde{\gamma}\Omega x);\ h_{16}=ch(\gamma\Omega y)\cdot\cos(\tilde{\gamma}\Omega x). \end{array}\right. \tag{5d}$$

Write separately the group of solutions at $\gamma=0$ and at $\gamma=\sqrt{0.5}$. When $\gamma=0$ we get two sets of four solutions. They are:

$$\left[\begin{array}{l} f_{17}=\sin(\Omega x);\ f_{18}=\cos(\Omega x);\\ f_{19}=ch(\Omega x);\ f_{20}=sh(\Omega x);\\ h_{17}=\sin(\Omega y);\ h_{18}=\cos(\Omega y);\\ h_{19}=ch(\Omega y);\ h_{20}=sh(\Omega y). \end{array}\right. \tag{5e}$$

Accordingly, at $\gamma=\sqrt{0.5}$, two sets of 16 functions for f_i and h_i given by (5a–5d) are partially coincides (at $i=1-8$) and this should be accounted for in introducing the respected set of considered solutions:

$$g_{1-8}\left(x,y,\sqrt{0.5}\right)=f_{1-8}\left(x,y,\sqrt{0.5},\sqrt{0.5}\right)= \\ =h_{1-8}\left(x,y,\sqrt{0.5},\sqrt{0.5}\right); \tag{5f}$$

$$\left[\begin{array}{l} g_{9-16}\left(x,y,\sqrt{0.5}\right)=f_{9-16}\left(x,y,\sqrt{0.5},\sqrt{1.5}\right); \\ g_{17-24}\left(x,y,\sqrt{0.5}\right)=h_{9-16}\left(x,y,\sqrt{0.5},\sqrt{1.5}\right). \end{array}\right. \tag{5g}$$

The drawback of these functions is that they cannot satisfy the boundary conditions on the whole boundaries or their sections. They can be fulfilled only in discrete points. Thus, it is suggested to combine these solutions (functions) with method of collocation, i.e., to require the satisfaction of boundary conditions only in discrete points. There are two boundary conditions in each boundary point. So, the number of functions (5) used for the solution of particular task should be two times higher than the number of boundary collocation points. The idea and technique of solution will be explained in details in Chapter V.

Galerkin Method

The essence of Galerkin method consists in several steps.

1. The looking for solution of (1a) is presented as:

$$W\left(x,y\right)=\sum_{k=k_1}^{K}\sum_{m=m_1}^{K}B_{k,m}X_k\left(x\right)Y_m\left(y\right), \tag{6a}$$

where $B_{k,m}$ are unknown coefficients. So-called trial functions $\left\{X_k\left(x\right)\right\}\Big|_{k=k_1}^{K}$;$\left\{Y_m\left(y\right)\right\}\Big|_{m=m_1}^{K}$; (k_1, m_1, K are chosen integers), generally satisfy the boundary conditions but generally they are not the exact solutions of equation (1a).

2. Substitute (5a) into (1a), we have:

$$\sum_{k=k_1}^{K}\sum_{m=m_1}^{K}B_{k,m}\Phi_{k,m}(x,y)=\omega^2\sum_{k=k_1}^{K}\sum_{m=m_1}^{K}B_{k,m}P_{k,m}(x,y), \tag{6b}$$

where:

$$\left[\begin{array}{l} \Phi_{k,m}\left(x,y\right)=X_k^{(IV)}\left(x\right)Y_m\left(y\right)+ \\ \qquad +2X_k''\left(x\right)Y_m''\left(y\right)+X_k\left(x\right)Y_m^{(IV)}\left(y\right), \\ P_{k,m}\left(x,y\right)=X_k\left(x\right)Y_m\left(y\right). \end{array}\right. \tag{6c}$$

3. Multiply equation (6b) on trial functions $X_r\left(x\right)Y_q\left(y\right)$ at specified $\left(r,q\right)\in Z; r=\overline{k_1,K}, q=\overline{m_1,K}$ and integrate each equation over the whole area of plate. In this way the system of linear equations is obtained with respect of unknown $B_{k,m}$:

$$\sum_{k=k_1}^{K}\sum_{m=m_1}^{K}B_{k,m}\cdot\left[\Phi1_{k,m}^{r,q}-\omega^2\cdot P1_{k,m}^{r,q}\right]=0, \tag{6d}$$

where:

$$\left[\begin{array}{l} \Phi1_{k,m}^{r,q}=\iint\Phi_{k,m}\left(x,y\right)\cdot X_r\left(x\right)\cdot Y_q\left(y\right)dS, \\ P1_{k,m}^{r,q}=\iint P_{k,m}\left(x,y\right)\cdot X_r\left(x\right)\cdot Y_q\left(y\right)dS. \end{array}\right. \tag{6e}$$

The only nontrivial solution exists when determinant of the system (6e) is equal to zero. This allows to find the natural frequencies ω_i $(i=\overline{1,K})$.

Remind the peculiarities of application of exponential as well as beam functions in Galerkin method.

Exponential functions

Their application is described in details in works [15–17]. Introduce the following designations for exponential functions:

$$\Gamma_k\left(x,L_x\right)=\exp\left(\frac{kx}{L_x}\right),\ \Gamma_m\left(y,L_y\right)=\exp\left(\frac{my}{L_y}\right), \tag{7a}$$

where $k,m\in Z$, and L_x , L_y – are the scaling parameters which are comparable with dimensions of plate in given direction.

Construction of trial functions $X_k\left(x\right), Y_m\left(y\right)$

Taking into account that each trial function should satisfy 4 boundary conditions (2 on each side) take them as a sum of 5 consecutive exponential functions $\Gamma_k\left(x,L_x\right)$ and $\Gamma_m\left(y,L_y\right)$:

$$X_k\left(x\right)=\sum_{i=0}^{4}\alpha_{k,i}\Gamma_{2+k-i}\left(x,L_x\right), \tag{7b}$$

$$Y_m\left(y\right)=\sum_{j=0}^{4}\gamma_{m,j}\Gamma_{2+m-j}\left(y,L_y\right). \tag{7c}$$

In formulas (7b) and (7c) the coefficients with zeroes index i ($\alpha_{k,0}$ and $\gamma_{m,0}$) are taken to be 1, and all other coefficients $\alpha_{k,i}$, $\gamma_{m,j}$ $\left(i,j=\overline{1,4}\right)$ are determined from the boundary conditions.

The advantage of exponential functions is that they are easily analytically differentiated and integrated.

Beam functions

Beam functions $X_k\left(x\right)$ and $Y_m\left(y\right)$ are the linear combinations of Krylov's functions [19]:

$$\begin{cases} S_1(\beta z) = 0.5\left[ch(\beta z) + \cos(\beta z)\right]; \\ S_2(\beta z) = 0.5\left[sh(\beta z) + \sin(\beta z)\right]; \\ S_3(\beta z) = 0.5\left[ch(\beta z) - \cos(\beta z)\right]; \\ S_4(\beta z) = 0.5\left[sh(\beta z) - \sin(\beta z)\right]. \end{cases} \tag{8a}$$

In general cases functions $X_i(x)$ and $Y_j(y)\left(i, j = \overline{1, K}\right)$ are presented as:

$$X_i(x) = X_i(0)\cdot S_1\left(\frac{k_i x}{a}\right) + X_i'(0)\cdot S_2\left(\frac{k_i x}{a}\right) + \\ + X_i''(0)\cdot S_3\left(\frac{k_i x}{a}\right) + X_i'''(0)\cdot S_4\left(\frac{k_i x}{a}\right); \tag{8b}$$

$$Y_j(y) = Y_j(0)\cdot S_1\left(\frac{m_j y}{b}\right) + Y_j'(0)\cdot S_2\left(\frac{m_j y}{b}\right) + \\ + Y_j''(0)\cdot S_3\left(\frac{m_j y}{b}\right) + Y_j'''(0)\cdot S_4\left(\frac{m_j y}{b}\right). \tag{8c}$$

These functions are very simple in differentiating and slightly more complicated in integration, which requires special elaboration [14].

Four sides clamped rectangular plate

The intension of paper is a verification of new methods, so no new or unusual geometries will be considered here. So, four sides clamped rectangular isotropic plate $0 \le x \le a,\quad 0 \le y \le b$ is considered where a and b – are length and width of the plate. The boundary conditions are:

$$\begin{cases} W|_{x=0} = 0; \quad \left.\frac{\partial W}{\partial x}\right|_{x=0} = 0; \quad W|_{x=a} = 0; \quad \left.\frac{\partial W}{\partial x}\right|_{x=a} = 0; \\ W|_{y=0} = 0; \quad \left.\frac{\partial W}{\partial y}\right|_{y=0} = 0; \quad W|_{y=b} = 0; \quad \left.\frac{\partial W}{\partial y}\right|_{y=b} = 0. \end{cases} \tag{9a}$$

Trial functions in Galerkin method

The calculation of four unknown coefficients $\alpha_{k,i}$ and $\gamma_{m,j}\left(i, j = \overline{1,4}\right)$ in trial functions (7b), (7c) is presented in details in work [15]. This is reduced to solutions of four linear equations.

When the trial functions are the beam functions the substitution of conditions (8a) into general presentation (8b) gives the following expressions for them:

$$X_i(x) = S_3\left(\frac{k_i x}{a}\right) + \frac{S_3(k_i)}{S_4(k_i)}\cdot S_4\left(\frac{k_i x}{a}\right), \tag{9b}$$

where k_i are solution of the transcendental equation:

$$ch(k_i)\cdot\cos(k_i) = 1. \tag{9c}$$

The roots of (9c) $k_i\left(i = \overline{1,K}\right)$ can be calculated only by numerical methods, which is the possible source of error. Here, for example we give the following first three roots of (9c): $k_1 = 4.300407449$; $k_2 = 7.853204624$; $k_3 = 10.99560784$.

Collocation boundary conditions in method of SES

We treat the boundary $P = P_1 \cup P_2 \cup P_3 \cup P_4$ as a set of discrete points. In given case each side of plate $\Lambda_k\ \left(k = \overline{1,4}\right)$ is substituted by N points (Fig. 1):

$$\begin{aligned} P_{1,j} &= \left\{(x, y) : x = x_j = \frac{a(j-1)}{N} + \frac{a}{2N};\ y = 0\right\}; \\ P_{2,j} &= \left\{(x, y) : x = x_j = \frac{a(j-1)}{N} + \frac{a}{2N} + \varepsilon_x;\ y = b\right\}; \\ P_{3,j} &= \left\{(x, y) : x = 0;\ y = y_j = \frac{b(j-1)}{N} + \frac{b}{2N}\right\}; \\ P_{4,j} &= \left\{(x, y) : x = a;\ y = y_j = \frac{b(j-1)}{N} + \frac{b}{2N} + \varepsilon_y\right\},\ j = \overline{1, N}. \end{aligned} \tag{10}$$

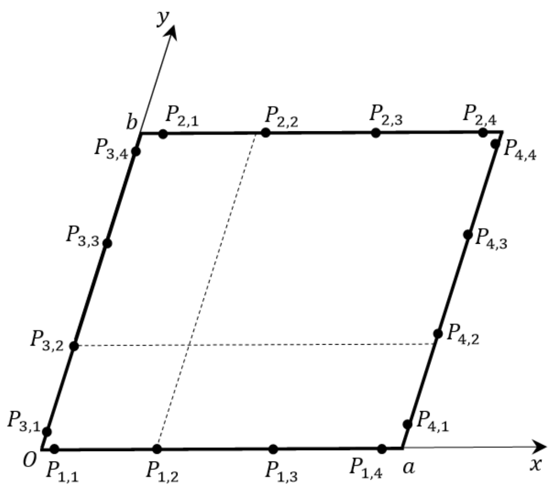

Fig. 1. Asymmetrical placement of points

The parameters ε_x and ε_y are the small disturbances as compared with the distances between the adjacent point. They are introduced artificially here, because due to the geometrical symmetry of task and symmetry of choice of functions $f_i\left(\Omega, x, y, \gamma, \hat{\gamma}\right)$ and $h_i\left(\Omega, x, y, \gamma, \hat{\gamma}\right)$ the system of equations can degenerate. Two boundary conditions should satisfy in each of $4N$ boundary points, so at whole we have $8N$ boundary conditions.

Variants of numerical realizations

In Galerkin method we take the same number of functions K with respect to x and y, which give the general dimension of the resulting system of equation. So, the choice of K functions means K^2 equations. As to SES method N points on each sides means $8N$ equations (unknowns).

The object of investigation is the accuracy of each method as to the number of unknowns. Yet in SES method the selection of the particular Voigt solution might have influence on the accuracy. The same can be said on the particular placement of the collocation points. In this work we formulated the corresponding rules in advance and do not tried other variants except the described below.

So, in method of SES the general solution of (1a) is presented as follows:

$$W(x,y)=\sum_{m=1}^{8N} b_m\cdot F_m(\Omega,x,y,\gamma), \qquad (11)$$

where b_m are unknown coefficients, $F_m(\Omega,x,y,\gamma)$ are some subjectively chosen functions compiled from f_i, h_i and g_i, see formulas (5). For $N=2$ it is necessary to chose 16 functions $F_m(\Omega,x,y,k)$, for $N=3$ we need 24 functions F_m, for $N=4-32$ functions F_m and so on.

Results and their analysis

Investigation on the convergency of results on example of first frequency

The results for first frequency for two different rectangular plates are presented in Table 1. All three techniques (SES and two variants of Galerkin method) were employed.

Make some explanations as to the choice of Voigt functions in SES method. For $N=2$ the number of functions should be equal to 16, and we choose the following ones: $\{f_{17}-f_{20}, h_{17}-h_{20}, g_1-g_8\}$. At $N=4$ we must take 32 functions. So, they are the same as in case $N=2$ plus 16 functions $\{f_1-f_8, h_1-h_8\}$, given by (5a) and (5b) at $\gamma=\sqrt{0.2}$.

Adding two more points on each side requires employment of 16 additional functions. They are chosen as functions $\{f_1-f_8, h_1-h_8\}$ for some other value of $\gamma, 0<\gamma<\sqrt{0.5}$. All employed values of γ^2 for various number of the side points are given in Table 1. As to parameters ε_x and ε_y at calculation they were chosen to be the following: $\varepsilon_x=a/(20N)$ and $\varepsilon_y=b/(20N)$.

As it follows from the Table 1 both Galerkin methods

Table 1. The values of first natural frequency ω_1 with respect to the number of unknowns

Method of SES			Galerkin method		
$8N$, number of equations	The values of γ^2 in equations (5a) and (5b)	Calculated frequency	K^2, number of equations	Exponential functions	Beam functions
a/b = 1					
16	–	35,289007	1	36,0109	36,108678
32	0,2	36,019756	9	35,989827	36,006762
48	0,1; 0,2	35,988377	25	35,985464	35,991484
64	0,1; 0,2; 0,3	35,987131	49	35,985217	35,987547
80	0,1; 0,2; 0,3; 0,4	35,985446	81	35,985193	35,986225
96	0,05; 0,1; 0,2; 0,3; 0,4	35,985291	121	–	35,985699
112	0,05; 0,1; 0,15; 0,2; 0,3; 0,4	35,985221	169	–	35,985463
a/b = 1,5					
16	–	59,718587	1	60,903124	60,957063
32	0,2	60,804412	9	60,769458	60,796478
48	0,1; 0,2	60,765354	25	60,761871	60,771733
64	0,1; 0,2; 0,3	60,761256	49	60,761179	60,765224
80	0,1; 0,2; 0,3; 0,4	60,761248	81	60,761108	60,762968
96	0,05; 0,1; 0,2; 0,3; 0,4	60,761175	121	–	60,762043
112	0,05; 0,1; 0,15; 0,2; 0,3; 0,4	60,761121	169	–	60,761617

demonstrate a good convergence even at very small number of unknowns. This is because the trial functions exactly satisfy the boundary conditions for this task. Exponential functions lead to better accuracy as compared with beam functions and this might justify their preferable application for more complicated geometries. The considered task with clamped edges is not a favorable case as to show the efficiency of MSES. Nevertheless, the increase of the number of unknowns in it eventually leads to the little better accuracy as to the beam function method. Yet the salient advantage of the MSES is that it be can easily applied for any geometry and for any boundary conditions.

Comparison with the literature results

The comparison with the literature results is given in Table 2 and Table 3. Our results are presented for the maximal number of equations employed for each particular method. Table 2 contains the results for square plate. As it follows from it the exponential functions based GM, EFBGM, demonstrates the very impressive accuracy.

The employment of only 81 terms gives practically the identical results with those of Liu and Banerjee [22] and El-Gamel et al. [25] where much more equations were used. This impressive agreement (with 7 to 8 digit) is attained for all six consecutive natural frequencies. From other hand, in work of Liu and Banerjee it was stated [22] that machine accuracy was actually attained. So, it has no sense to increase the number of equations in EFBGM. As to beam functions based GM, BFBGM, it also shows the good accuracy, however, this method is inferior in accuracy even for twice larger number of equations. The method of SES generally demonstrates the similar accuracy as BFBGM, even for smaller number of equations and even for this unfavorable boundary conditions.

The similar conclusions can be drawn for rectangular plate with length ratio equal to 1.5. EFBGM demonstrates the perfect accuracy even for relatively smaller number of terms. For us it is important that MSES still has a very good accuracy. As to BFBGM it is close to the results of MSES.

Conclusions

1. The principally new method of SES for plate vibration based on fundamental solution of Voigt is suggested. In contrast to similar known methods, it employs the frequency dependent functions for both space coordinates. The boundary conditions are fulfilled only in chosen collocation points, which makes the method very simple, versatile for any geometry and boundary conditions. It easily can be adjusted for complex structures consisting of many structural elements.

2. The verification of method of SES is performed for rectangular plate with the most unfavorable fully clamped boundary conditions. Nevertheless, the results are very encouraging and show good correspondence with other methods, at least, for first 6 frequencies. Note, that

Table 2. First six frequencies for square plate

Frequency number	Method of SES, 112 equations	Galerkin method		Liu and Banerjee [22]	El-Gamel et al,
		Exponential functions, 81 equations	Beam functions, 169 equations		
1	35,985221	35,985193	35,985463	35,98519	35,985191
2	73,394328	73,393877	73,394955	73,39385	73,393857
3	73,394328	73,393877	73,394955	73,39385	73,393857
4	108,21816	108,21671	108,22220	108,2165	108,21652
5	131,58077	131,58078	131,58303	131,5808	–
6	132,20523	132,20490	132,20731	132,2048	–

Table 3. First six frequencies for rectangular plate with $a/b = 1,5$

Frequency number	Method of SES, 112 equations	Galerkin method		Sakata, [26]	El-Gamel et al., [25]
		Exponential functions, 81 equations	Beam functions, 169 equations		
1	60,761121	60,761108	60,761616	60,761099	60,761139
2	93,834006	93,833588	93,835714	93,833474	93,833784
3	148,78117	148,77978	148,78199	148,77973	148,77994
4	149,67389	149,67438	149,67780	149,67424	149,68349
5	179,56336	179,56166	179,57260	179,56110	–
6	226,82655	226,86414	226,83612	–	–

Voigt functions were chosen subjectively, almost at random, without detailed consideration of the best combination of particular functions and placement of point of collocation. This testify that even for this geometry the better results might be achieved by this method.

3. Two variants of the Galerkin method are realized and compared. First one, EFBGM, where trial functions employ exponential functions. As in work [17] it leads to superior accuracy, is very effective and can be applied to treatments of differential equations of very different task. As to the beam functions based GM, it is reliable method, but, in spite of very wide application, it still is inferior to EFBGM. Besides it can hardly be generalized for differential equations for other structure, say, for thick plates.

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Застосування вибіркових наборів точних розв'язків Фойта для задачі коливань тонких пластин

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Анотація. *Для дослідження коливань пластин запропоновано принципово новий метод вибіркових точних розв'язків (МВТР), який ґрунтується на основі фундаментальних розв'язків Фойгта. На відміну від аналогічних відомих методів, МВТР використовує функції, що залежать від шуканої частоти для обох просторових координат. Побудовано множини часткових точних розв'язків основного диференціального рівняння, які залежать від певних параметрів, обраних довільним чином. Це дозволяє вибрати будь-яку кількість точних розв'язків, причому їх необхідна кількість залежить від граничних умов, які мають задовольнятися в точках колокації. Продемонстровано ефективність методу для найбільш несприятливого випадку закріплення пластини – її жорсткого защемлення по всіх сторонах. Тим не менш, точність методу є цілком задовільною для визначених перших власних частот навіть при відносно невеликій кількості вузлів колокації. Це свідчить про широкі перспективи застосування запропонованого методу для складних конструкцій, різної геометрії, різноманітних граничних умов. Для дослідження збіжності й точності МВТР додатково реалізовано та порівняно два варіанти методу Гальоркіна, перший з яких використовує експоненціальні функції, а другий – дуже популярні балочні функції. Результати розрахунків показують перевагу МВТР як в технічній реалізації, так і в точності, що свідчить про можливість його подальшого застосуваннях в більш складних динамічних задачах будівельної механіки.*

Ключові слова: *прямокутна пластина, вільні коливання, жорстко защемлена пластина, метод Гальоркіна, розв'язки Фойгта.*