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Semi-analytical implicit direct time integration method for 1-D gas dynamic problem

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Abstract. *Sharp wave treatment for 1-D gas dynamic problem is still a challenge for modern numerical methods. They often require too many space and time steps, produce spurious oscillation of solution, exhibit a strong numerical dissipation or divergence of results. This paper is further extension of authors' idea of employment the analytical solution for space coordinate, where time step is a parameter which used in the space solution. Its peculiarity consists in development of additional linearization procedure of dependence between the pressure and density. It is performed in premise that actual pressure for each space element is close to the basic pressure, attained at previous moment of time. The efficiency of method is tested on the very popular task of Sod, where two different ideal gases in a tube are separated by diaphragm, which is suddenly broken. The problem considered in Lagrangian coordinates formulation. The results obtained show the very good method efficiency, which requires the essentially lesser time and space steps, leads to no spurious oscillation and give consistent and predictable results with respect to meshing. The accuracy of method is mostly controlled by time step, which should be larger than clearly stated theoretical lower limit. Other advantage of method is that it can calculate the process to any desired time moment, and space meshing can be variable in time and space and can be easily adapted during the process of calculation.*

Keywords: *transfer matrix method, Lagrangian formulation, implicit method, Sod's task, stability, ideal gas.*

Introduction

Finite difference method in application to 1D gas dynamic problem was, seemingly, the first example of numerical integration of partial differential equations [1]. Its origin was necessitated by that the traditional analytical techniques such as method of characteristics or Fourier method of variables separation were suitable only for linear problems such as linear hydrodynamic or solid deformable bodies. For gas the main variables are interrelated by highly nonlinear equations of state, which complicates the analytical treatment.

Most finite difference methods were ineffective as to treatment of problems with discontinuities and produced spurious oscillations near the shock front. Von Neumann and Richtmyer [2] inserted in numerical scheme the artificial viscosity term which somehow averaged the positions of several points. Lax [3] suggested the triangular scheme, which was not suitable for the small time steps and employ only linearly distributed initial states. Godunov's method, known as finite volume method [4] was the important step development of the numerical methods in general and was based on the integral form of the method of characteristics.

The state of art of these early methods was outlined in work [5]. Among the very detailed consideration of the most popular and effective numerical schemes, this work is remarkable by that it gives the theoretically elaborated exact solution for the shock tube problem. In this task two gases in different initial states were separated by diaphragm, which is suddenly completely broken, and this induces the shock waves in both parts. This task becomes the effective test benchmark for many subsequent numerical schemes.

All above methods were explicit ones, where the state of the given point at the next time moments was simply calculated from the states of some neighboring

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points at previous moment. This is explained by the shortage of computational power at that time. The explicit methods are only conditionally stable and their accuracy is provided only within short time range.

The modern FD methods [6,7] and finite volume methods [8] are mostly implicit. It means that resulting FDM equations contain several unknown parameters at the next moment of time and calculation can be performed only for the whole piping system as solution of the equations system. The implicit methods are intendent as for slow as for fast transient processes and they allow to considerer complex multibranched geometries. The high computational cost of implicit methods can be partly compensated by implementation of the adaptive time step and spatial grid strategy [9] depending on the relative rate of dynamic process.

The development of methods for numerical treatment of 1D gas dynamic problems is still a relevant issue. They are applied for examination of: the impact of the hydrogen blended natural gas on the linepack energy under emergency scenarios of the pipeline operation [10]; for treatment of dynamic processes for very complicated marine pipelines with application of new enhanced state equation of highly compresses natural gas [11]; for modeling of transient processes in a gas during the local leak [12] and global fracture of pipeline [13].

In recent decades, the wide attention has attracted the simulation of 1D pressure waves in the in nuclear reactor system [14]. They can be induced by: the rapid closing or opening of system devices, such as pump-valve [15]; steam bubble collapse [16]; loss-of-coolant accident (LOCA) due to leak in piping system [17].

The most known tool of treatment of transients in compressive media in nuclear power plant is the software RELAP, which is developed in Idaho National Laboratory. In its resent version, RELAP-7, the governing differential equations are discretized using a continuous Galerkin finite element method, and the resulting discrete equations are stabilized by artificial viscosity called the Entropy Viscosity Method [18]. Its accuracy is also demonstrated among other by comparison with solution of Sod [5].

In Electricite De France the specialized software Europlexus is developed during the long time. It is based on finite volume method using the arbitrary Lagrangian–Eulerian formulation for the resolution of the fluid problem and is able to take into account for the abrupt changes in area and junction of complex piping complex junctions [19]. The comprehensive assessment of accuracy and peculiarities of the method application [19] was performed in subsequent work [20], where among other the Sod task was analyzed.

The goal of this work is further elaboration of our semi analytical scheme [21], but now in application to gas dynamic problem. Accounting for the principal novelty of the method it will be verified only on example of the Sod's task, so many equations are considered in simplified form only. The aim is to show that our method is perfectly suitable for description of steep wave front – actually this task is the most complicated one in gas dynamic problem. The task is considered in Lagrangian formulation.

Governing equations

Start from the governing equations of the gas dynamic problem. Contrary to [21] analysis of behavior of the rod, where only two parameters were used, here 4 parameters, which characterize the state of the gas in any point, are used. Other difficulty, as compared with [21], lies in nonlinearity of the governing equations, which need to be linearized in calculation process.

Write down two basic differential equations in Lagrangian coordinates, which are related with each particular gas point (elementary mass) rather than with given point of pipeline, as used in Euler coordinates.

Start from equation of flow continuity (deformation). Consider the element of mass of the gas, Δm . Initially it occupies some volume V_0 and confined within the pipe length section Δx

$$\Delta m = \rho_0 V_0 , \tag{1a}$$

$$V_0 = S_0 \Delta x , \tag{1b}$$

where S_0 – is the area of pipe section, where the given mass was initially placed, ρ_0 is the density of the gas in initial state, and Δx is the difference between the initial position of the right and left borders of this mass. During the transient process the left border of this mass is located in space point with absolute coordinate $U_l(t)$, and the right side of it – in point $U_r(t)$ where t is the moment of time. The change of volume of this element Δm is given by formula:

$$\Delta V(t) = \left(L_r(t) - L_l(t)\right) S - \Delta x S_0 , \tag{2a}$$

here S is the area of pipe section, where the element Δm is located at considered moment of time. Introduce the notion of volumetric deformation, ε

$$\varepsilon = \frac{\Delta V(t)}{V_0} = \frac{\left(L_r(t) - L_l(t)\right) S - \Delta x S_0}{S_0 \Delta x} = \frac{\left(U_r(t) - U_l(t)\right) S - \Delta x S_0}{S_0 \Delta x} . \tag{2b}$$

If the area is constant along the whole pipeline, then deformation becomes:

$$\varepsilon = \frac{\Delta V(t)}{V_0} = \frac{\left(L_r(t) - L_l(t)\right) S - \Delta x S_0}{S_0 \Delta x} = \frac{\left(U_r(t) - U_l(t)\right) S - \Delta x S_0}{S_0 \Delta x} \to \frac{dU\left(x,t\right)}{dx} . \tag{2b}$$

Where $U(x,t)$ is the displacement (change of position) of any gas material point. On the other hand, from physical consideration we know, that the volumetric deformation of gas is proportional to the gas density in given state, $\rho(x)$. So, equating the physical notion of deformation and its geometrical definition (2b) we get:

$$\frac{dU}{dx} = \frac{\rho_0 - \rho(x,t)}{\rho(x)}. \tag{2c}$$

Equation (2c) is equation of continuity. It is the first main differential equation of gas dynamic task.

Second equation is the force equilibrium (momentum) equation. To get it, note that inertial force of elementary mass Δm should be equilibrated by gain of inner pressure ΔP:

$$-\Delta P - \rho_0 \Delta x \frac{d^2U}{dt^2} = 0. \tag{3}$$

Differential equations for gas flow (2c) and (3) are called in literature as hydraulic equations [22]. They mainly are responsible for the dynamical phenomena and gas flow. So, historically the researchers paid more attention for them, while assuming the temperature to be the constant or consider the adiabatic processes [6, 23]. For non-isothermal processes, where the heat exchange take place, very often the task is broken on two constituents – initially the hydraulic task is solved, for example for pressure and velocity, and then other two main parameters are determined from equation of state and the equations of conservation of energy [24, 25].

In simplified form, for ideal gas without thermal exchange with environment, these two equations are the following:

$$\rho = \frac{P}{RT}, \tag{4a}$$

$$\frac{T}{T_0} = \left(\frac{P}{P_0}\right)^{\frac{k-1}{k}}, \tag{4b}$$

where T is the temperature, R is the universal gas constant, and k is adiabatic constant for given gas. Substituting the temperature T from (4a) into energy equation (4b), we get:

$$\frac{\rho}{\rho_0} = \left(\frac{P}{P_0}\right)^{\frac{1}{k}}. \tag{4c}$$

Now we are able to write the exact mathematical formulation of the given problem. Reformulate the equation of continuity (2c) by considering for relationship (4c):

$$\frac{dU(x)}{dx} = \left(\frac{P_0}{P(x)}\right)^{\frac{1}{k}} - 1. \tag{5a}$$

So, now we have two governing differential equations (3) and (5a) both written in terms of displacement and pressure. But equation (5a) is nonlinear one. To linearize it present the inner pressure $P(x)$ for each considered mass element as the sum of some basic constant value P_b and correction function $P_s(x)$:

$$P(x) = P_b + P_s(x). \tag{5b}$$

It is assumed that basic value is much larger than the correction one:

$$P_b > P_s(x). \tag{5c}$$

Thus, inserting (5b) into (5a), obtain:

$$\frac{dU}{dx} + 1 = \left(\frac{P_0}{P_b}\right)^{\frac{1}{k}} \left(1 + \frac{P_s(x)}{P_b}\right)^{-1/k} \approx \alpha_1 - \alpha_2 P_s(x). \tag{5d}$$

Where the following coefficients are introduced:

$$\left(\frac{P_0}{P_b}\right)^{\frac{1}{k}} = \alpha_1; \qquad \left(\frac{P_0}{P_b}\right)^{\frac{1}{k}} \frac{1}{k P_b} = \alpha_2. \tag{5e}$$

For the sake of convenience, rewrite two governing differential equations together:

$$\frac{dU(x)}{dx} = (\alpha_1 - 1) - \alpha_2 P_s(x); \tag{6a}$$

$$\frac{dP_s(x)}{dx} = -\rho_0 \frac{d^2U(x)}{dt^2}. \tag{6b}$$

Take the derivative by x from first equation and combine these two equations to exclude the correction pressure $P_s(x)$. These two equations with two variables $P(x)$ and $U(x)$ can be reduced to one differential equations in partial derivatives with respect to x and t, which resembles very much that considered by us for the task of rod impact:

$$\frac{d^2U}{dx^2} - \alpha_2 \rho_0 \frac{d^2U}{dt^2} = 0. \tag{6c}$$

Description of semi analytical method for 1-D gas dynamic

The technique of solution is very similar to that for the impact rod problem [21].

The gas as material body is divided on small elementary masses, Δm^j, where $1 \le j \le J$. For the case of equal pipe area, it is reduced to the division of the whole initial length of gas on the sections $\Delta x^j = l^j$. Here we will consider that initial lengths of each elementary mass (sections) are the same, but this is not necessarily for the method ap-

plication, and will be discussed later. All subsequent equations are common for each element, and usually we will omit the upper index of space element " j ".

Consider the equal intervals of time Δt. Considered discrete moments of time are given by the following expression:

$$t_i = \Delta t \cdot i \quad (i = 0,1,...,i,i+1,...). \tag{7a}$$

So, all other parameters in the given moments of time will contain the lower indexes i. Apply the simplest finite difference scheme with respect to time integration of $\frac{d^2U}{dt^2}$ term – the central scheme, which is centered in the moment of time equal to t_{i-1}, then equation (6c) is written as:

$$\frac{d^2U_i(x)}{dx^2} - (c_{i-1})^2 \frac{U_i(x) - 2U_{i-1}(x) + U_{i-2}(x)}{\Delta t^2} = 0. \tag{7b}$$

Where:

$$(c_{i-1})^2 = \alpha_{2,i-1} \cdot \rho_0. \tag{7c}$$

So, within each element $0 \le x \le l^j$ the displacement is considered as continuous function of local length coordinate x. Then equation (7b) can be rewritten as:

$$\frac{d^2U_i(x)}{dx^2} - b^2 U_i(x) = -b^2\left(2U_{i-1}(x) - U_{i-2}(x)\right) = -b^2 Z_{i-1}(x). \tag{7d}$$

Where for convenience we introduced the special designation for the right side of equation:

$$Z_{i-1}(x) = 2U_{i-1}(x) - U_{i-2}(x) \tag{7e}$$

and constant c_{i-1}:

$$(c_{i-1})^2 t^{-2} = b^2. \tag{7f}$$

To solve the differential equation (7d) we need to specify its right-side, i.e. present function $Z_{i-1}(x)$ as the polynomial expansion up to second order:

$$Z_{i-1}(x) = c_0^{i-1} + c_1^{i-1} \cdot x + c_2^{i-1} \cdot x^2. \tag{8a}$$

Which eventually allows to obtain give the following solution for two main parameters of the problem within each elementary mass (section) [21]:

$$U_i(x) = U_{i,0}K_1(x) + \alpha_2\left(P_b - P_{i,0}\right)K_2(x) + c_0^{i-1}\left(1 - K_1(x)\right) + c_1^{i-1}\left(x - K_2(x)\right) + c_2^{i-1}\left(x^2 + \frac{2}{b^2}\left(1 - K_1(x)\right)\right) + (\alpha_1 - 1)K_2(x); \tag{8b}$$

$$P_i(x) = -U_{i,0}\frac{b^2K_2(x)}{\alpha_2} + P_{i,0}K_1(x) + P_b\left(1 - K_1(x)\right) + c_0^{i-1}\frac{b^2K_2(x)}{\alpha_2} + \frac{\alpha_1 - 1}{\alpha_2}\left(1 - K_1(x)\right) - \frac{c_1^{i-1}}{\alpha_2}\left(1 - K_1(x)\right) - \frac{2c_2^{i-1}}{\alpha_2}\left(x - K_2(x)\right). \tag{8c}$$

Where functions $K_1(x)$ and $K_2(x)$ are so-called generalized Krylov's functions:

$$K_1(x) = ch(xb); \quad K_2(x) = sh(xb)/b. \tag{8d}$$

Which have the remarkable properties:

$$K_1'(x) = b^2K_2(x); \quad K_2'(x) = K_1(x). \tag{8e}$$

The solutions (8b) and (8c) are given in form suitable for application of transfer matrix method, which is very popular in analysis of 1D problems [26], especially in structural mechanics [27]. Application of formulas (8b) and (8c) for each elementary mass section Δx, application of conjugation (continuity) equations at the border between any neighboring mass sections:

$$U_i^j\left(l^j\right) = U_i^{j+1}(0); \quad P_i^j\left(l^j\right) = P_i^{j+1}(0) \tag{8f}$$

and boundary conditions at both outer boundaries, allow to complete the mathematical statement of problem at given moment of time.

To go to next moment of time we need to perform two additional procedures. First to find new function $Z_i(x)$ (8a), and, second, to choose new basic value of pressure P_b. Present $U_i(x)$ as

$$U_i(x) = \alpha_i K_1(x) + \beta_i K_2(x) + D_i(x). \tag{9a}$$

Where $D_i(x)$ is second order polynomial. The values of coefficients α_i, β_i and polynomial $D_i(x)$ can easily be got by comparing (8b) with (9a) [21]. Next step is expansion of Krylov's functions into polynomial series. So, we get $U_i(x)$ as second order polynomial. The last step in getting $Z_i(x)$ is to find the linear combination (7e) of expanded now $U_i(x)$ with previously expanded $U_{i-1}(x)$.

As to choosing of P_b with respect to which the linearization is performed. Similarly as to $U_i(x)$ in (9a), present the solution for pressure $P_i(x)$ (8c) in form:

$$P_i(x) = \alpha_{i,P}K_1(x) + \beta_{i,P}K_2(x) + D_{i,P}(x). \tag{9b}$$

We can define P_b for the next time step $i+1$ as the mean value of $P_i(x)$ (9b), thus:

$$P_{b,i+1} = \frac{1}{\Delta x}\int_0^{\Delta x}\left(\alpha_{i,P}K_1(x) + \beta_{i,P}K_2(x) + D_{i,P}(x)\right)dx. \tag{9c}$$

Taking into account the expressions for Krylov's functions (8d) and polynomial terms in (8c), we get:

$$P_{b,i+1} = \frac{\alpha_{i,P} \cdot b \cdot sh\left(l^j \cdot b\right) + \beta_{i,P}\left(ch\left(l^j \cdot b\right) - 1\right)}{b^2 l^j} + P_{b,i} + \frac{\alpha_1 - 1}{\alpha_2} - \frac{c_1^{i-1}}{\alpha_2} - \frac{\left(c_2^{i-1}\right)^2}{2\alpha_2} l^j . \quad (9d)$$

So, all theoretical background is outlined and we can calculate and analyze the real examples.

Analysis of the Sod's task

The initial data are given on Fig. 1 [5, 20].

Left-side high-pressure chamber $P = 1\,bar, \quad \rho = 1\,\frac{kg}{m^3}$		Right-side low-pressure chamber $P = 0.1\,bar, \quad \rho = 0.125\,\frac{kg}{m^3}$		
$x = 0\,m$		$x = 0.5\,m$		$x = 1\,m$

Fig. 1. Sketch of the single-phase gas shock-tube with closed $(U = 0)$ boundary conditions at the inlet $(x = 0\,m)$ and at the outlet $(x = 1\,m)$ of the pipe

The pipe of length 1m consists of two equal (0.5 m) chambers separated by diaphragm. The left-side chamber contains the ideal gas at $P = 1\,bar$ and density $\rho = 1\,\frac{kg}{m^3}$. The initial parameters for left-side gas are $P = 0.1\,bar$ and $\rho = 0.125\,\frac{kg}{m^3}$. The ideal gas is characterized by adiabatic constant $k = 1.4$. At initial (zero) time moment the diaphragm is broken, and transient process starts. So, the task is the description of the wave propagation process along the tube.

Here we will use the previous finding [21] as to the method stability; the spurious oscillation might occur when the time step is lower than allowable. Due to employment of three term polynomial expansion of function $Z_{i-1}(x)$ it is expected that minimal allowable time step should satisfy the following inequality for each element of mass j [21]:

$$l^j b^j \le 5 . \quad (10a)$$

Accounting for expression for b:

$$\Delta t \ge \frac{l^j}{5}\sqrt{\left(\frac{P_0}{P_b}\right)^{\frac{1}{k}} \frac{1}{kP_b}\rho_0} = \frac{l^j}{5}\sqrt{\frac{\rho_0}{kP_0}}\sqrt{\left(\frac{P_0}{P_b}\right)^{\frac{1}{k}+1}} . \quad (10b)$$

So, the stability of method depends not only from initial meshing, which is controlled by factor of initial state – $\sqrt{\frac{\rho_0}{kP_0}}$.

Stability also depends from possible rarefaction. For example, if the basic (mean) pressure at some moment become 5 time smaller than that at the beginning, than minimum time interval should be at least 4 times larger, than was initially established. Note, that in such cases it is possible to make more fine meshing as to space element lengths. The semi-analytical method is very easily adjustable to length modifications.

Make additional comment as to the method stability as compared with the most popular ones. The usual requirement in literature is that [19]:

$$\Delta t \le 0.8 \cdot l^j \cdot \sqrt{\left(\frac{P_0}{P_b}\right)^{\frac{1}{k}} \frac{1}{kP_b}\rho_0} . \quad (10c)$$

So, the requirements (10b) and (10c) are quate opposite – our method is more stable at larger time interval, while other methods require lesser time intervals.

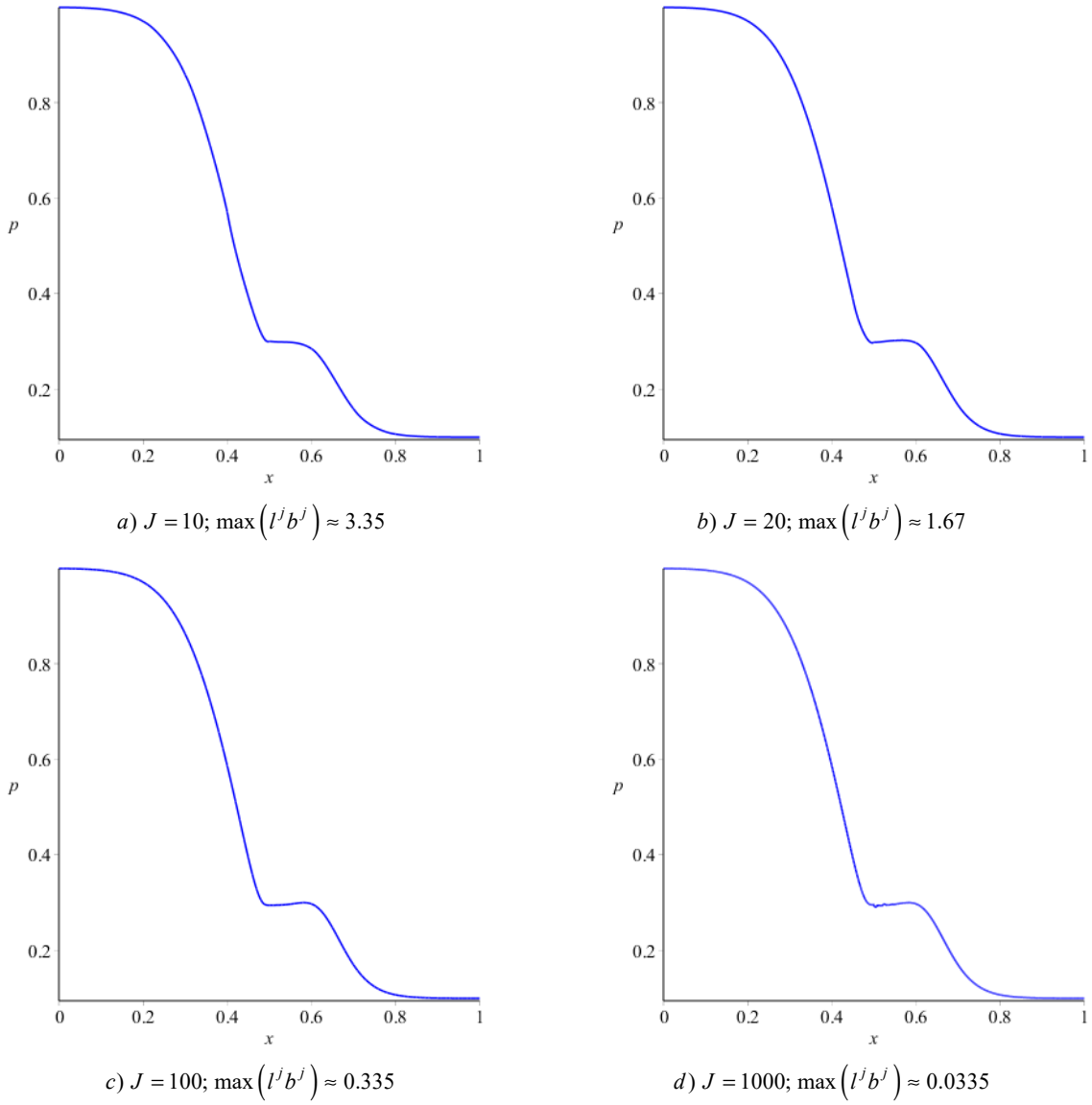
As in work [21] demonstrate that main meshing parameter which controls the accuracy is the time interval, while the length of element, provided it satisfies the requirement of stability (10b), has lesser influence on accuracy. Choose the moment of time $T_0 = 4.5125 \cdot 10^{-4}\,c$, which was considered in works [5, 20], and calculate the distribution of pressure along the whole pipe.

Show that length of element has the lesser importance on accuracy of results. Fix the time interval Δt such as to arrive for the considered time moment T_0, in five time steps, i.e.:

$$\Delta t = \tau_s = \frac{T_0}{s} . \quad (11a)$$

Consider 4 different space meshing, which are characterized by number J, of evenly spaced elements: $a)\ J = 10,\ l^j = \frac{1}{10}m;\ b)\ J = 20,\ l^j = \frac{1}{20}m;\ c)\ J = 100;$ $d)\ J = 1000$. The results of calculation are shown on Fig. 2. As one can see, the number of space elements has almost no influence on the calculated results. In all these cases the requirement of stability was fulfilled at initial moment of time $(t = 0)$, and the corresponding maximal values of $l^j b^j$ for all elements are given on the figures. At subsequent time moments the values of P_b become smaller, so $\max(l^j b^j)$ for $J = 10$ may become around 7 at the time of interest T_0. Nevertheless, visually the results for $J = 10$ are stable.

Now demonstrate that time step interval is main meshing parameters which controls the accuracy of results. Fix the number of space elements $J = 500$. The different time intervals will be chosen to arrive to time moment T_0 an



a) $J=10$; $\max\left(l^j b^j\right)\approx 3.35$ *b*) $J=20$; $\max\left(l^j b^j\right)\approx 1.67$

c) $J=100$; $\max\left(l^j b^j\right)\approx 0.335$ *d*) $J=1000$; $\max\left(l^j b^j\right)\approx 0.0335$

Fig 2. The results of calculation of pressure at the moment of time $T_0=4.5125\cdot 10^{-4}\ c$ for the same time step $\Delta t=\frac{T_0}{5}$, but for different number of space elements: *a*) $J=10$; *b*) $J=20$; *c*) $J=100$; *d*) $J=1000$

integer number of time steps. For convenience, introduce the notion of unitary time step τ:

$$\tau=\frac{T_0}{20}. \tag{11b}$$

And take the following intervals of time: $\Delta t\approx\left\{2\tau;\frac{2}{3}\tau;\right.$ $\left.0.4\tau;0.2\tau;0.1\tau\right\}$, which means that corresponding number, ii, of time steps are: $ii=\{10,30,50,100,200\}$. The results of calculations (except for $ii=100$) are shown on Fig 3. Fig. 3 clearly demonstrates the prevailing effect of the time step on the results accuracy. The results for $ii=50$ are very close to such for $ii=200$ and results for $ii=100$ (not shown here) are visually coincide with those for $ii=200$.

Note, that value of the stability control parameter $l^j b^j$ at $ii=200$ was 2.4–2.68. It means, that at $ii=100$ it is equal to 1.2–1.34. In its turn, it means that we might apply as small as only $J=150$ space elements for number of time step equal to $ii=100$. Note, the similar accuracy was attained in work [19] at 1000–2000 space elements (see Fig. 2 *a* in [20]). All this testify about the very good accuracy of the method.

Note, that all existing numerical and theoretical result are restricted to the time moment when the wave front reaches the pipe wall. Our method is not restricted by similar conditions. So, consider the further extension of wave beyond the time T_0. Fix the number of space elements, take $J=500$. All further results are shown for this space meshing.

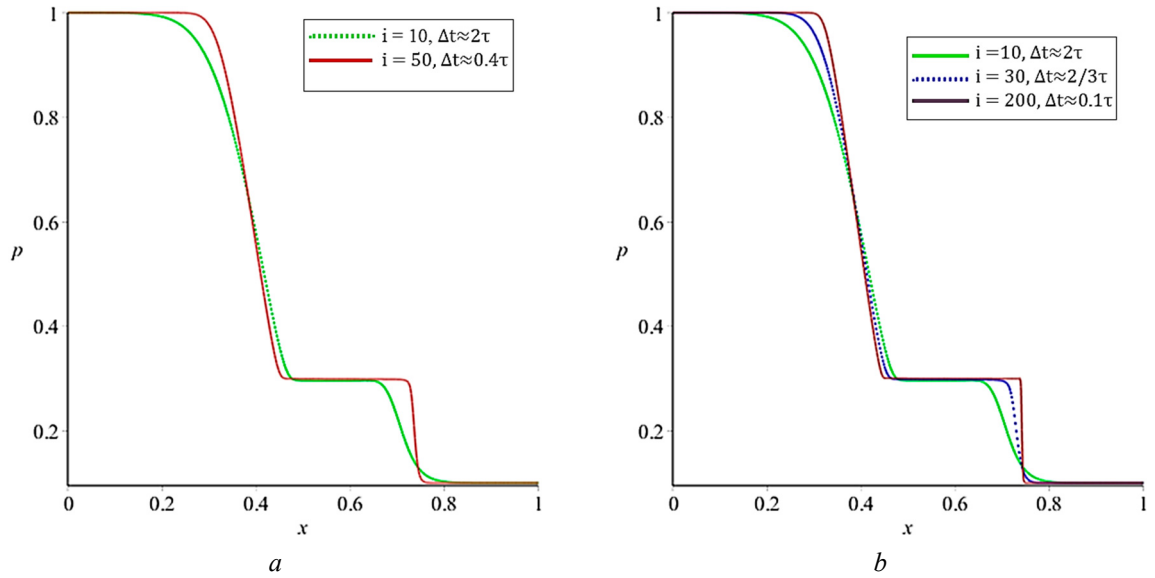


Fig. 3. The results of pressure calculation at the moment of time $T_0 = 4.5125 \cdot 10^{-4}\ c$ for the same space element number $J = 500$ but for different number of time steps, ii: *a*) $ii = \{10, 50\}$; *b*) $ii = \{10, 30, 200\}$

Take the time step fixed too:

$$\tau_1 = \frac{\tau}{5} = \frac{T_0}{100}. \qquad (11c)$$

Apart from "usual" time moment $t = T_0$, consider two additional moments of time: $t = 1.5T_0$ and $t = 2T_0$. This means that calculations are performed for $ii = \{100, 150, 200\}$ time steps. The results are shown on Fig. 4. The results are logical, clear and produce no spurious oscillations. The left wave slope decreases with time, and this agrees with numerical and theoretical results of work [17].

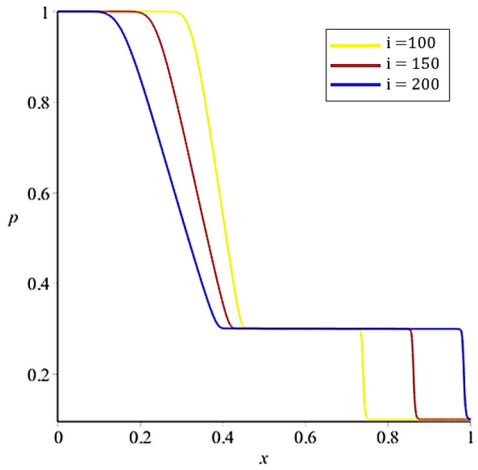


Fig 4. The results of pressure calculation for the same number of elements, time step interval, but for different time: *a*) for $ii = 100$, $t = T_0$; *b*) for $ii = 150$, $t = 1.5T_0$; *c*) for $ii = 200, t = 2T_0$

Continue the calculations of pressure for larger time moments, the parameters of space and time meshing being the same as above. Now consider the moments of time after the reflection of the wave from the side boundary of pipe, Fig. 5.

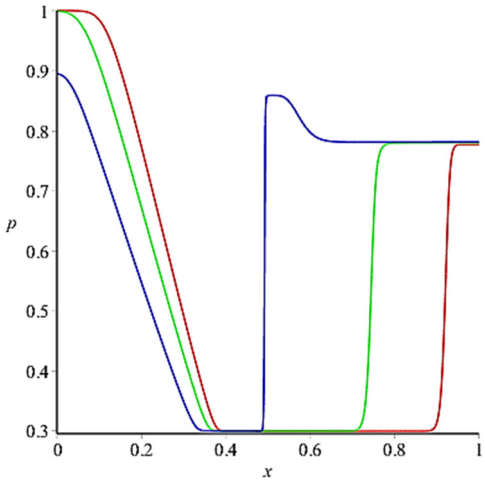


Fig. 5. The results of pressure calculations for the whole pipe at number of space element $J = 500$; time step $\tau_1 = \frac{T_0}{100}$ for different time moments: *a*) $t = 2.2 \cdot T_0$ (red line); *b*) $t = 2.5 \cdot T_0$ (green); *c*) $t = 3 \cdot T_0$ (blue)

The first graph (red line) is shown for the time $t = 2.2 \cdot T_0$, i.e., at 220th time step. The next graph (green line) is obtained for $t = 2.5 \cdot T_0$ i.e., at 250th time step, and third graph is calculated for $t = 3 \cdot T_0$, $ii = 300$. We can not

comment the accuracy of them, but the graphs look as the consistent ones, the straight lines keep their straightness, there are no visual oscillations, which are usually exhibited in numerical methods.

Conclusions

Semi-analytical method is applied for 1D gas dynamic problem formulated in Lagrangian coordinates. Generally, the method resembles the one for the rod impact problem, but nonlinear dependence between the pressure and density requires the development of additional linearization procedure. It is performed in assumption that the looking for pressure is close to the assumed basic pressure, which is determined as a mean value from the previous time step. Technically, the realization of the method is performed by transfer matrix method, which make it easy to code. The efficiency of method is tested on the task of Sod, and it is demonstrated that:

1. The results of calculation are very consistent, they produce no spurious oscillations, and they correspond very well with theoretical finding and calculations by other numerical scheme. The results are stable, provided that time step is larger than minimum admissible one, and this principally differ it from existing ones, where maximal time step is usually restricted.

2. The time step is the main parameter of meshing which controls the accuracy. So, the different space meshing give the same results, if the time step is the same. In contrary, if the space step is fixed the results can be essentially improved by employing the lesser time steps. The method allows to calculate the transient effects as before as well as after the reflection of wave from the boundary.

3. Employment the analytical solution withing each space element as well as realization of transfer matrix method allow to simplify the calculation procedure. It can be made very flexible, where the length of each element can independently be dynamically adjusted for the attained state of this element.

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Напіваналітичний неявний метод інтегрування по часу одномірної газодинамічної задачі

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Анотація. *Метод зважених нев’язок набув широкої популярності протягом останніх років, особливо завдяки застосуванню в методах скінчених елементів. Він полягає в наближеному виконанні диференціальних рівнянь, тоді як граничні умови мають виконуватись точно. Ця мета досягається правильним вибором множин пробних (базових) функцій, які дають нев’язки. Нев’язки множать на вагові функції та мінімізують, інтегруючи по всій області задачі. Множина пробних і вагових функцій визначає особливість та переваги кожного конкретного методу. Найбільш популярним є вибір пробних і вагових функцій у вигляді тригонометричних або поліноміальних функцій. У двовимірних задачах часто використовуються так звані “балочні функції”, які є рішеннями більш простих одновимірних задач для балки.*

В даній методичній роботі ми досліджуємо можливість використання множин функцій, побудованих на послідовних експоненціальних функціях, які точно задовольняють граничним умовам. Метод досліджено на прикладі простої осесиметричної задачі оболонки, точне рішення якої відоме для будь-якого навантаження. Для кількох прикладів розподіленого або концентрованого навантаження запропонований метод порівнюється з аналогічним методом Нав'є, в якому використовуються тригонометричні функції. Також ретельно досліджується правильний вибір вагових функцій. Зазначається, що запропоновані множини симетричних чи антисиметричних експоненціальних функцій мають хорошу перспективу для застосування в більш складних задачах структурної механіки

Ключові слова: *осесиметрична оболонка, розподілене навантаження, концентрована сила, метод Нав’є, метод Бубнова-Гальоркіна, множина експоненціальних функцій.*