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Method of matched sections in application to thin-walled and Mindlin rectangular plates

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Abstract. *The paper elaborates the principally new variant of finite element method in application to plate problem. It differs from classical FEM approach by, at least, three points. First, it uses the strong differential formulation rather than the weak one and suppose the approximate analytical solution of all differential equations. Second, it explicitly uses all geometrical and physical parameters in the procedure of solution, rather than some chosen ones, for example, displacement and angles of rotation as usually done in FEM formulation. Third, the conjugation between adjacent elements occurs between the adjacent sections rather than in polygon vertexes. These conditions require the continuity of displacements, angles, moments and forces. Each side of rectangular elements is characterized by 6 main parameters, so, at whole there are 24 parameters for each rectangular element. The right and upper sides' parameters are considered as output ones, and they are related with lower and left sides ones by matrix equations, which allows to apply transfer matrix method for the compilation of the resulting system of equations for the whole plate. The numerical examples for the thin-walled and Mindlin plates show the high efficiency and accuracy of the method.*

Keywords: *Method of matched sections, transfer matrix method, rectangular plate, Mindlin plate, boundary conditions, conjugation.*

1. Introduction

Plates are the basic construction elements used in various industries, and its proper analysis is a requirement of many standards. Yet theory of plate has tremendous significance for the development of applied mathematics, in general, and partial differential equations, in particular. The solutions of plate bending problem led to the appearance of brilliant mathematical methods, some of which are now indispensable in modern mathematical physics. The history of plate theory is intriguing and fascinated [1] and is very instrumental in understanding of solution methods for partial differential equations, PDE [2].

The theoretical method of Navier was the first example of application of double trigonometrical (Fourier) series to PDE; while the Levi method was the first one in reduction of PDE to the ordinary differential equation when the looking for function is considered as product of known function of one coordinate (due to specific boundary conditions) by the unknown function of another coordinate [3]. These methods are considered as the exact ones, because each function of the general solution satisfies to the governing PDE. The exact methods are still popular in literature, yet their application is restricted to plate of particular form and type of loading [4], or they require the special procedure of treatment of rather simple boundary conditions [5].

The theory of plate has generated two versatile mathematical techniques both being proposed at the beginning of 20th century due to new challenges by ship building industry. They are – a) Ritz method [6]; b) Galerkin method, GM [7]. Both methods are approximate ones. GM operates by set of products of unknown coefficients on known trial functions which satisfy to boundary conditions. This set is inserted into resulting PDE, which gives so-called residual. The residual is consequently multiplied on each one trial function and integrated over the domain area, which give the corresponding system of linear equations. The generalization of Galerkin method gave rise to more general weighed residual methods [7]. Ritz method also operates by similar functions with unknown coefficients which are substituted in the energy functional (instead of governing

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PDE as in GM) over the domain area. It contains the products of pairs of unknown coefficients. The minimization of functional gives the required system of equations. Both methods are very widely used in mathematical physics [8], in general, and are popular in plate analysis up to now [6, 9, 10, 11], in particular.

All above (semi-) analytical methods employ the functions which are continuous over the whole domain. They imply the cumbersome theoretical manipulation. Yet their main drawback is inability to handle problems with complicated boundaries, variable properties (initial, or changed due to complicated physical behavior) and geometrical form, thickness, inclusions, cut-outs and so on. In this sense, the methods which operates by local functions defined within the local volumes, have indispensable advantages.

The finite element method stemmed out from the Ritz method, where the whole domain is considered as assemblage of simple geometrical shapes (elements), separated by nodes. The known specially constructed interpolation functions (with the property of partition of unit) are defined within these small volumes. Within each element the solution is presented as a sum of products of interpolation functions on the unknown geometrical positions of the nodes. In the treatment of plate, the most popular are rectangular elements, which contains 12 known interpolation functions and 12 unknown node positions: each of four nodes is characterized by displacement and two (in x direction and y direction) angles of rotation [12]. To provide better continuity between elements the more complicated interpolation functions are used, which accounts for larger number of assumed degrees of freedom at nodes. For example, it was suggested to employ additional fourth degree of freedom at each node – mixed derivative with respect to both coordinates [13].

FEM is very popular method; it is used in overwhelming majority of engineering application. As to plate analysis it sometimes suffers from the locking phenomena for relatively thick (Mindlin) plate. This problem manifests itself as an overly stiff system when the plate thickness tends to zero, and related with inability of the interpolation functions to be able to represent the Kirchoff plate behavior [14]. Other drawback of conventional FEM plate element is its conjugation with different structural elements, for example, beam element. This requires the development and justification of special variational procedures [15, 16] to avoid the spurious stress between the interface of plate and beam elements.

Our goal is to develop a principally new FE approach for plate analysis. In contrast to conventional one, where the neighboring elements are conjugated only at nodes and only by some chosen in advance degrees of freedom (say, displacements and angles of rotation), our elements conjugates at neighboring sides and by all six degrees of freedom, which completely characterize the section as a beam. They are: displacement, two angles of rotation – normal to the section and tangential ones, two moments – normal and tangential (twisting) ones, and transverse force. So, in contrast to the conventional (nodes matched) FEM (or NM-FEM), our method can be called as (section matched) SM-FEM, or for brevity – Method of Matched Sections, MMS. Other difference between conventional FEM and MMS is that relations between kinematic and force parameters (so-called stiffness matrix) in NM-FEM is not evident and derived from variational principle (minimization of functional), while in MMS the relationship is evident, is a beam-like one and is derived directly from physical equations of dependence of strains from stresses.

2. Differential equations and their solution

2.1. Differential equations for the Mindlin plate

The peculiarity of our solution is that we do not combine all partial differential equations together to form one governing equation with respect to one main parameter of the problem – we analyze and approximately solve each differential equation separately. So, we need to write them.

Consider the rectangular element, Fig 1, with sides equal to a (along x direction) and b (along y direction). All parameters and their positive directions are shown on Fig 1. Start from the force equilibrium equation:

$$\frac{\partial Q_y(x,y)}{\partial y}+\frac{\partial Q_x(x,y)}{\partial x}=p(x,y). \tag{1a}$$

Where Q_x and Q_y are the transverse shear forces, $p(x,y)$ is outer distributed loading, as shown on Fig. 1*a*.

Consider the moment equilibria around y axis:

$$Q_x=\frac{\partial M_x(x,y)}{\partial x}+\frac{\partial M_\tau^y(x,y)}{\partial y}. \tag{1b}$$

And around x axis:

$$Q_y=\frac{\partial M_y(x,y)}{\partial y}+\frac{\partial M_\tau^x(x,y)}{\partial x}. \tag{1c}$$

Where M_x and M_y are normal bending moment as shown on Fig. 1*a*, M_τ^y and M_τ^x are twisting bending moment and upper indexes indicate the plane of their application, Fig. 1*a*.

Next step is compilation of the physical equations. For normal bending moments we have [12]:

$$M_x=D\left(\frac{\partial \theta_x(x,y)}{\partial x}+\nu\frac{\partial \theta_y(x,y)}{\partial y}\right); \tag{2a}$$

$$M_y=D\left(\frac{\partial \theta_y(x,y)}{\partial y}+\nu\frac{\partial \theta_x(x,y)}{\partial x}\right). \tag{2b}$$

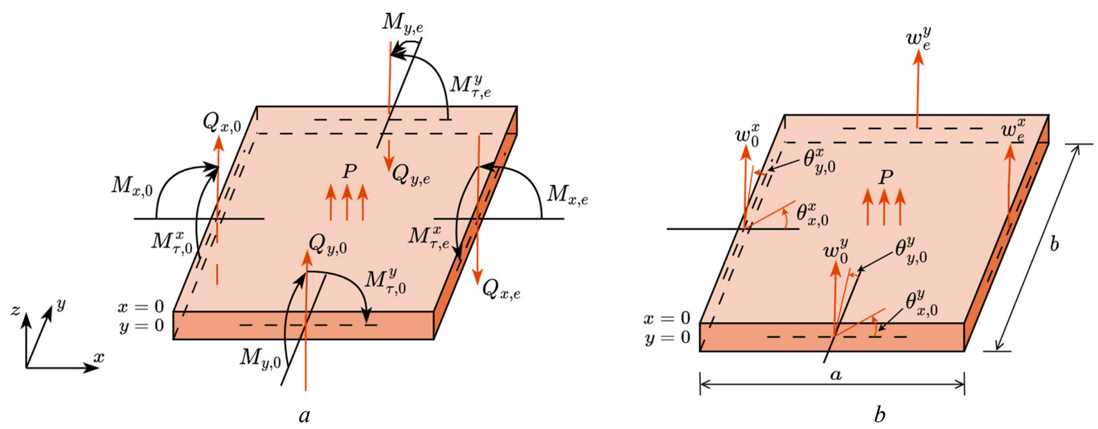

Fig. 1. The general scheme of rectangular plate element: *a*) directions of force parameters, *b*) directions of kinematic parameters

Where v is Poisson ratio, θ_x and θ_y are the angles of rotation of the normal to the middle surface of plate, Fig. 1*b*, and D is flexural rigidity of the plate:

$$D = \frac{Eh \cdot h^2}{12\left(1-v^2\right)}. \tag{2c}$$

Where h is the thickness of plate. For the twisting moments it can be written:

$$M^x_\tau = \frac{Gh \cdot h^2}{12} \frac{2\partial\theta_y}{\partial x} = D(1-v)\frac{\partial\theta_y}{\partial x}; \tag{2d}$$

$$M^y_\tau = D(1-v)\frac{\partial\theta_x}{\partial y}. \tag{2e}$$

Where we used the known dependence between the constants for isotropic material:

$$G = \frac{E}{2(1+v)}. \tag{2f}$$

The last set of governing equations are the geometrical equations with shear force correction as in Timoshenko beam. The gain of displacement w in x direction, is:

$$\frac{\partial w(x,y)}{\partial x} = \theta_x(x,y) + \gamma_x(x,y). \tag{3a}$$

Where the notion of the shear angle γ_x is introduced, namely it makes the difference between thin-walled plate and a Mindlin plate. The value of γ_x is proportional to the shear transverse force Q_x, as given by the following:

$$\gamma_x = -\frac{6Q_x}{5hG}. \tag{3b}$$

In similar way write for gain of displacement w in y direction

$$\frac{\partial w}{\partial y} = \theta_y + \gamma_y. \tag{3c}$$

Where:

$$\gamma_y = -\frac{6Q_y}{5hG}. \tag{3d}$$

2.2. Idea of method

The idea of method is inspired by analogy with the beam. The plate element is considered as two beams, one extending from the middle of the left side to the middle of the right side (in x direction), and second one is a beam extending from the middle of lower side to the upper side of plate (in y direction). X beam is characterized by 6 parameters, which depend only on x. All these parameters are related to the central line of this beam $\left(x, \frac{b}{2}\right)$. They are:

– transverse displacement $w^x(x)$, where superscript shows affiliation to the X beam;

– angle of rotation of the beam $\theta^x_x(x)$, and angle of twisting of the beam $\theta^x_y(x)$;

– distributed moments applied to X beam: $M_x(x)$ – bending moment, which leads to the gain of angle of rotation of beam sections, and twisting moment $M^x_\tau(x)$ which leads to the gain of angle of twisting;

– distributed transverse force $Q_x(x)$.

Analogously, Y beam is also characterized by 6 parameters, which depend only on y, and they pertain to the central line of this beam $\left(\frac{a}{2}, y\right)$. They are: displacement $w^y(y)$; angle of rotation of the beam $\theta^y_y(x)$, and angle of twisting of the beam $\theta^y_x(y)$; distributed bending $M_y(y)$ and twisting $M^y_\tau(y)$ moments; distributed transverse force $Q_y(y)$.

These 12 parameters change along the corresponding axes according to above differential equations (1), (2), (3). Solve them and find the dependences of the parameters along the central lines.

2.3. Procedure of solution of plate equations

According to the main idea of two beam-like interacting solutions, rewrite the differential equation (1a), as:

$$\frac{\partial Q_y(y)}{\partial y}+\frac{\partial Q_x(x)}{\partial x}=P. \tag{4a}$$

Where for each small element we can take that $p(x,y)=P$. Assume that:

$$\frac{\partial Q_x(x)}{\partial x}=const=A_1. \tag{4b}$$

Where A_1 is a constant to be determined later. Then, we get from (4b), that:

$$Q_x(x)=Q_{x,0}+A_1x. \tag{4c}$$

And from (4a) and (4b), that:

$$Q_y(y)=Q_{y,0}+(P-A_1)y. \tag{4d}$$

Where lower index "0" indicate the affiliation to the beginning section of the corresponding beam, Fig. 1.

Let us go up. Rewrite the differential equations with respect to moments according to the "beam" approximation:

$$\frac{\partial M_x(x)}{\partial x}=Q_x(x)-\frac{\partial M_\tau^y(y)}{\partial y}; \tag{5a}$$

$$\frac{\partial M_y(y)}{\partial y}=Q_y(y)-\frac{\partial M_\tau^x(x)}{\partial x}. \tag{5b}$$

There are two still unknown functions in each of equations (5a) and (5b). So, we need to introduce two additional unknown constants A_2, and A_3 for determination of twisting moments:

$$\frac{\partial M_\tau^y(y)}{\partial y}=A_2=const; \tag{5c}$$

$$\frac{\partial M_\tau^x(x)}{\partial x}=A_3=const. \tag{5d}$$

Where from we get the approximate expressions for twisting moments:

$$M_\tau^y(y)=M_{\tau,0}^y+A_2y; \tag{5e}$$

$$M_\tau^x(x)=M_{\tau,0}^x+A_3x. \tag{5f}$$

Where as in above, the lower indexes "0" means that value pertains to the beginning section of the "beam". Availability of twisting moments allows to find the bending moments. So, the integration of (5a) and (5b) gives:

$$M_x(x)=M_{x,0}+Q_{x,0}x+A_1\frac{x^2}{2}-A_2x; \tag{6a}$$

$$M_y(y)=M_{y,0}+Q_{y,0}y+(P-A_1)\frac{y^2}{2}-A_3y. \tag{6b}$$

By the way, note that the values of bending moments as integrally overaged (lover index "ovg") or found at the center of each "beam" line (lover index "cnt") will be used in subsequent calculation. Keeping in mind that $0\le x\le a$ and $0\le y\le b$, Fig. 1, we get for the values of moments in the middle points:

$$M_{x,cnt}=M_x\left(\frac{a}{2}\right)=M_{x,0}+Q_{x,0}\frac{a}{2}+A_1\frac{a^2}{8}-A_2\frac{a}{2}; \tag{6c}$$

$$M_{y,cnt}=M_y\left(\frac{b}{2}\right)=M_{y,0}+Q_{y,0}\frac{b}{2}+$$
$$+(P-A_1)\frac{b^2}{8}-A_3\frac{b}{2}. \tag{6d}$$

In the similar way write the integrally overaged values of bending moments along the whole length of each "beam". They are the following:

$$M_{x,ovg}=\frac{1}{a}\int_0^a M_x(x)dx=M_{x,0}+$$
$$+Q_{x,0}\frac{a}{2}+A_1\frac{a^2}{6}-A_2\frac{a}{2}; \tag{6e}$$

$$M_{y,ovg}=\frac{1}{b}\int_0^b M_y(y)dy=M_{y,0}+Q_{y,0}\frac{b}{2}+$$
$$+(P-A_1)\frac{b^2}{6}-A_3\frac{b}{2} \tag{6f}$$

Further proceed from already determined parameters to still undefined ones. Consider the physical equations for the twisting angles. According to adopted "beam" model they are rewritten from (2d) and (2e) in the form:

$$\frac{\partial\theta_y^x(x)}{\partial x}=\frac{M_\tau^x(x)}{D(1-\nu)} \tag{7a}$$

$$\frac{\partial\theta_x^y(y)}{\partial y}=\frac{M_\tau^y(y)}{D(1-\nu)} \tag{7b}$$

Their integration gives the following expressions:

$$\theta_y^x(x)=\theta_{y,0}^x+\frac{M_{\tau,0}^x}{D(1-\nu)}x+\frac{A_3}{D(1-\nu)}\frac{x^2}{2} \tag{7c}$$

$$\theta_x^y(y)=\theta_{x,0}^y+\frac{M_{\tau,0}^y}{D(1-\nu)}y+\frac{A_2}{D(1-\nu)}\frac{y^2}{2} \tag{7d}$$

Now consider the bending angles. Rewrite (2a) and (2b):

$$\frac{\partial\theta_x(x)}{\partial x}=\frac{M_x(x)}{D(1-\nu^2)}-\nu\frac{M_y(y)}{D(1-\nu^2)}; \tag{8a}$$

$$\frac{\partial\theta_y(y)}{\partial y}=\frac{M_y(y)}{D(1-\nu^2)}-\nu\frac{M_x(x)}{D(1-\nu^2)}. \tag{8b}$$

It is impossible to integrate directly these equations, because the right sides depend on two different variables. So, two options are possible. One is to take "alien" moments in the central points, and other – as overaged values. In first case we get the following expressions for the bending angles:

$$\theta_x^x(x)=\theta_{x,0}^x+\frac{M_{x,0}x+Q_{x,0}\frac{x^2}{2}+A_1\frac{x^3}{6}-A_2\frac{x^2}{2}}{D(1-\nu^2)}-$$
$$-\nu\frac{M_{y,0}+Q_{y,0}\frac{b}{2}+(P-A_1)\frac{b^2}{8}-A_3\frac{b}{2}}{D(1-\nu^2)}\cdot x; \tag{8c}$$

$$\theta_y^y(y)=\theta_{y,0}^y+\frac{M_{y,0}y+Q_{y,0}\frac{y^2}{2}+(P-A_1)\frac{y^3}{6}-A_3\frac{y^2}{2}}{D(1-\nu^2)}-$$
$$-\nu\frac{M_{x,0}+Q_{x,0}\frac{a}{2}+A_1\frac{a^2}{8}-A_2\frac{a}{2}}{D(1-\nu^2)}\cdot y. \tag{8d}$$

In second case, the expressions for bending angles are slightly different:

$$\theta_x^x(x)=\theta_{x,0}^x+\frac{M_{x,0}x+Q_{x,0}\frac{x^2}{2}+A_1\frac{x^3}{6}-A_2\frac{x^2}{2}}{D(1-\nu^2)}-$$
$$-\nu\frac{M_{y,0}+Q_{y,0}\frac{b}{2}+(P-A_1)\frac{b^2}{6}-A_3\frac{b}{2}}{D(1-\nu^2)}\cdot x; \tag{8e}$$

$$\theta_y^y(y)=\theta_{y,0}^y+\frac{M_{y,0}y+Q_{y,0}\frac{y^2}{2}+(P-A_1)\frac{y^3}{6}-A_3\frac{y^2}{2}}{D(1-\nu^2)}-$$
$$-\nu\frac{M_{x,0}+Q_{x,0}\frac{a}{2}+A_1\frac{a^2}{6}-A_2\frac{a}{2}}{D(1-\nu^2)}\cdot y. \tag{8f}$$

And now proceed to the last step in integration of the plate equations. Get the expressions for displacements. Rewrite the dependences (3), as below:

$$\frac{\partial w^x(x)}{\partial x}=\theta_x^x(x)-\frac{6Q_x(x)}{5hG}; \tag{9a}$$

$$\frac{\partial w^y(y)}{\partial y}=\theta_y^y(y)-\frac{6Q_y(y)}{5hG}. \tag{9b}$$

Their integration leads to long but rather simple formulas:

$$w^x(x)=w_0^x+\theta_{x,0}^x x+\frac{M_{x,0}\frac{x^2}{2}+Q_{x,0}\frac{x^3}{6}+A_1\frac{x^4}{24}-A_2\frac{x^3}{6}}{D(1-\nu^2)}-$$
$$-\nu\frac{M_{y,0}+Q_{y,0}\frac{b}{2}+(P-A_1)\frac{b^2}{8}-A_3\frac{b}{2}}{D(1-\nu^2)}\cdot\frac{x^2}{2}-\frac{6}{5hG}\left(Q_{x,0}x+A_1\frac{x^2}{2}\right). \tag{9c}$$

$$w^y(y)=w_0^y+\theta_{y,0}^y y+\frac{M_{y,0}\frac{y^2}{2}+Q_{y,0}\frac{y^3}{6}+(P-A_1)\frac{y^4}{24}-A_3\frac{y^3}{6}}{D(1-\nu^2)}-$$
$$-\nu\frac{M_{x,0}+Q_{x,0}\frac{a}{2}+A_1\frac{a^2}{8}-A_2\frac{a}{2}}{D(1-\nu^2)}\cdot\frac{y^2}{2}-\frac{6}{5hG}\left(Q_{y,0}y+(P-A_1)\frac{y^2}{2}\right). \tag{9d}$$

Where the "alien" moments are taken in the central points. Alternatively, we get the following formulas:

$$w^x(x)=w_0^x+\theta_{x,0}^x x+\frac{M_{x,0}\frac{x^2}{2}+Q_{x,0}\frac{x^3}{6}+A_1\frac{x^4}{24}-A_2\frac{x^3}{6}}{D(1-\nu^2)}-$$
$$-\nu\frac{M_{y,0}+Q_{y,0}\frac{b}{2}+(P-A_1)\frac{b^2}{6}-A_3\frac{b}{2}}{D(1-\nu^2)}\cdot\frac{x^2}{2}-\frac{6}{5hG}\left(Q_{x,0}x+A_1\frac{x^2}{2}\right). \tag{9e}$$

$$w^y(y)=w_0^y+\theta_{y,0}^y y+\frac{M_{y,0}\frac{y^2}{2}+Q_{y,0}\frac{y^3}{6}+(P-A_1)\frac{y^4}{24}-A_3\frac{y^3}{6}}{D(1-\nu^2)}-$$
$$-\nu\frac{M_{x,0}+Q_{x,0}\frac{a}{2}+A_1\frac{a^2}{6}-A_2\frac{a}{2}}{D(1-\nu^2)}\cdot\frac{y^2}{2}-\frac{6}{5hG}\left(Q_{y,0}y+(P-A_1)\frac{y^2}{2}\right). \tag{9f}$$

when the 'alien" moments are taken as overaged values.

3. Methodology of solution

3.1. Field transfer matrix for element

A few words about transfer matrix method, TMM. It is most suitable for 1-D problems and was thoroughly described in [17]. It is very popular for one dimensional problems, in particular for solution of spatial beams. A lot of practical application of TMM is given in [18]. It is very instrumental in organization of the calculational process, and we will use some of its ideas in our calculations.

Turn out to our plate solution. As it follows from derived solution (4)–(9), to calculate all parameters in any section of either of two "beams" we need to have 15 constants: 6 main parameters of *X*-beam, 6 parameters of *Y*-beam, and 3 auxiliary constants - A_1, A_2, and A_3. These 3 auxiliary constants are found from 3 additional conditions of coupling of two beams. They are the requirement of equality of displacement and angles of rotations in two "beams" in the central point:

$$w^x\left(\frac{a}{2}\right)=w^y\left(\frac{b}{2}\right); \tag{10a}$$

$$\theta_x^x\left(\frac{a}{2}\right)=\theta_x^y\left(\frac{b}{2}\right); \tag{10b}$$

$$\theta_y^x\left(\frac{a}{2}\right)=\theta_y^y\left(\frac{b}{2}\right). \tag{10c}$$

In essence, these equations show the mechanism of "welding" together two independent beams. Equations (10) allows to express the coefficients A_1, A_2, and A_3 through the 12 main parameters (initial conditions at the left and lower sides of the plate (beginning of *X*-beam and beginning of *Y*-beam). Schematically the solution of above equations (10) can be presented in the matrix form:

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}=\begin{Bmatrix} \alpha_{1,1};\alpha_{1,2};\ldots\alpha_{1,11};\alpha_{1,12} \\ \alpha_{2,1};\alpha_{2,2};\ldots\alpha_{1,11};\alpha_{2,12} \\ \alpha_{3,1};\alpha_{3,2};\ldots\alpha_{3,12};\alpha_{3,12} \end{Bmatrix}\begin{pmatrix} Z_{1,0} \\ Z_{2,0} \\ \ldots \\ Z_{12,0} \end{pmatrix}+P\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}. \tag{10d}$$

Where all coefficients $\alpha_{i,j}$ and β_i are known. Also in (10d) we for conveniency renumerate all 12 initial conditions as:

$$w_0^x=Z_1;\quad \theta_{x,0}^x=Z_2;\quad \theta_{y,0}^x=Z_3;$$
$$M_{x,0}=Z_4;\quad M_{\tau,0}^x=Z_5;\quad Q_{x,0}=Z_6. \tag{11a}$$

$$w_0^y=Z_7;\quad \theta_{y,0}^y=Z_8;\quad \theta_{x,0}^y=Z_9;$$
$$M_{y,0}=Z_{10};\quad M_{\tau,0}^y=Z_{11};\quad Q_{y,0}=Z_{12}. \tag{11b}$$

Then we are able to separately compile the equations for all six parameters, which characterize *X*-beam in each section $x=const$, they formally can be presented as:

$$\begin{pmatrix} Z_1(x) \\ Z_2(x) \\ \ldots \\ Z_6(x) \end{pmatrix}=\begin{Bmatrix} a_{1,1}(x);a_{1,2}(x);\ldots a_{1,11}(x);a_{1,12}(x) \\ a_{2,1}(x);a_{2,2}(x);\ldots a_{1,11}(x);a_{2,12}(x) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{6,1}(x);a_{6,2}(x);\ldots a_{6,12}(x);a_{6,12}(x) \end{Bmatrix}\begin{pmatrix} Z_{1,0} \\ Z_{2,0} \\ \ldots \\ Z_{12,0} \end{pmatrix}+$$
$$+P\begin{pmatrix} b_1(x) \\ b_2(x) \\ \ldots \\ b_6(x) \end{pmatrix}. \tag{12a}$$

Where for example, $Z_1(x)=w^x(x)$, $Z_6(x)=Q_x(x)$ (enumerated as in (11a)). Evidently, the coefficients are such that $a_{m,m}(0)=1$, and all other coefficients are equal to zero at point $x=0$, i.e., $a_{m,k}(0)=0$ and all $b_m(0)=0$. Analogously for six parameters, which characterize the state of *Y*-beam, it can be written:

$$\begin{pmatrix} Z_7(y) \\ Z_8(y) \\ \ldots \\ Z_{12}(y) \end{pmatrix}=\begin{Bmatrix} c_{1,1}(y);c_{1,2}(y);\ldots c_{1,11}(y);c_{1,12}(y) \\ c_{2,1}(y);c_{2,2}(y);\ldots c_{1,11}(y);c_{2,12}(y) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{6,1}(y);c_{6,2}(y);\ldots c_{6,12}(y);c_{6,12}(y) \end{Bmatrix}\begin{pmatrix} Z_{1,0} \\ Z_{2,0} \\ \ldots \\ Z_{12,0} \end{pmatrix}+$$
$$+P\begin{pmatrix} d_1(y) \\ d_2(y) \\ \ldots \\ d_6(y) \end{pmatrix} \tag{12b}$$

Where for example, $Z_7(y)=w^y(y)$ (enumerated as in (11b)). Evidently, coefficients are such that $c_{m,m+6}(0)=1$, and all other coefficients are equal to zero at point $y=0$, i.e., $c_{m,k}(0)=0$ and all $d_m(0)=0$.

According to the logic of TMM it is convenient to formally specify the set of auxiliary additional 12 unknown constants (which according to TMM can be easily eliminated at further steps of calculation process). These constants are the values of main parameters of two "beams" at their ends. So additional parameters for *X*-beam are the following:

$$\begin{pmatrix} Z_{13} \\ Z_{14} \\ \ldots \\ Z_{18} \end{pmatrix}=\begin{Bmatrix} a_{1,1}(a);a_{1,2}(a);\ldots a_{1,11}(a);a_{1,12}(a) \\ a_{2,1}(a);a_{2,2}(a);\ldots a_{1,11}(a);a_{2,12}(a) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{6,1}(a);a_{6,2}(a);\ldots a_{6,12}(a);a_{6,12}(a) \end{Bmatrix}\begin{pmatrix} Z_{1,0} \\ Z_{2,0} \\ \ldots \\ Z_{12,0} \end{pmatrix}+$$
$$+P\begin{pmatrix} b_1(a) \\ b_2(a) \\ \ldots \\ b_6(a) \end{pmatrix}. \tag{13a}$$

Where for example, $Z_{13}=w^x(x=a)=w_e^x$, where the lower subscript "*e*" means the value of specific parameter at the end of the "beam". In the same way introduce the set of 6 additional parameters at the end of *Y*-beam:

$$\begin{pmatrix} Z_{19} \\ Z_{20} \\ \ldots \\ Z_{24} \end{pmatrix}=\begin{Bmatrix} c_{1,1}(b);c_{1,2}(b);\ldots c_{1,11}(b);c_{1,12}(b) \\ c_{2,1}(b);c_{2,2}(b);\ldots c_{1,11}(b);c_{2,12}(b) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{6,1}(b);c_{6,2}(b);\ldots c_{6,12}(b);c_{6,12}(b) \end{Bmatrix}\begin{pmatrix} Z_{1,0} \\ Z_{2,0} \\ \ldots \\ Z_{12,0} \end{pmatrix}+$$
$$+P\begin{pmatrix} d_1(b) \\ d_2(b) \\ \ldots \\ d_6(b) \end{pmatrix}. \tag{13b}$$

Where for example, $Z_{19} = w^y\left(y=b\right) = w_e^y$.

Equations (12) are termed as a field transfer matrix (or connection equations), which gives solution in any point $\left(x,y\right)$ by knowing the state at initial (inlet) point. In particular, when considered points are the outlet points (upper or right sides) of plate, then equations (13) give relation of state in them with the inlet parameters at the beginning of plate (left and lower sides).

3.2. Conjugation equations and the general algorithm for the whole plate

Let we have the "big" plate which is meshed by "small" elements, Fig. 2. First of all, we should enumerate the elements. In simplest case of rectangular plate, we have N rows and K columns, which are enumerated consequently from bottom to top and from left to right. So, traversal of elements is carried out from left to right with the subsequent transition to the next row. Thus, we adopt the following enumeration of elements:

$$m = k + n \cdot K, \quad 1 \le m \le KN. \tag{14a}$$

Where *m* is the number of element, *k* is number of column, *n* is number of row.

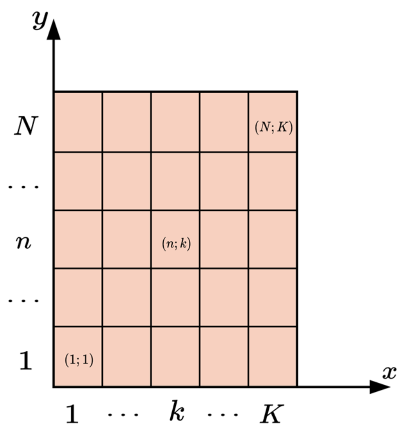

Fig. 2. The scheme of the meshing

Next step is enumeration of unknown parameters. Now for simplicity of explanation of the method we adopt that each element is characterized by 24 unknowns: 12 are inlet unknowns, and 12 are outlet ones, which are related with former by field transfer matrix (connection) equations. To underline that each unknown pertains to some particular element, introduce the upper additional indexes to them:

$$Z_t^{k,n} = Z_t^m = R_{t+24 \cdot m} = R_l, \quad 1 \le l \le L = 24 \cdot KM. \tag{14b}$$

Where $1 \le t \le 24$ is the inner numeration within each element, and R_l is particular designation of unknown for continuous numbering of unknowns.

Now consider the conjugation equations (point transfer matrixes). Consider particular element $\left(k,n\right)$. At its left side it borders with right side of element $\left(k-1;n\right)$ (the side is common for both). Evidently, at this side all 6 parameters of two "beams" are the same. So, we can write six conjugation equations between them:

$$Z_{13}^{k-1,n} = Z_1^{k,n}; \quad Z_{14}^{k-1,n} = Z_2^{k,n}; \ldots Z_{18}^{k-1,n} = Z_6^{k,n}. \tag{14c}$$

In the same way we can write 6 conjugation equations for element $\left(k,n\right)$ at its lower side:

$$Z_{19}^{k,n-1} = Z_7^{k,n}; \quad Z_{20}^{k,n-1} = Z_8^{k,n}; \ldots Z_{24}^{k,n-1} = Z_{12}^{k,n}. \tag{14d}$$

As we see, at each common side of two "beams" we have 6 equations. So, we can formally consider that for each side of each element we have 3 equations (6 equations for two sides). So, formally for all 4 border sides of each element we have 12 equations. If the particular side is situated on the boundary of the plate, then we have 3 boundary conditions here. So, formally the number of conjugation and boundary conditions for each element is always the same –12.

Thus, for each element we have 24 unknowns. We have 12 connection equations for them, as well as 12 conjugation and boundary conditions. Formally, the number of unknowns and number of equations are the same.

The big number of unknowns used here (equal to $24 \cdot KM$) is a not necessity in a practical MMS realization. It is given here mostly for the easiness of understanding of method. Actually, as we see, in each element the outlet parameters are actually redundant. This allows to decrease the number of unknowns by 2 times. Furthermore, in all inner elements the inlet parameters can be expressed through the inlet parameters of element placed at the left and bottom sides to them. Thus, so-called process of eliminations [17] of unknowns can be applied. Eventually, the number of unknowns theoretically can be reduced to six multiplied on the number of sides which are placed on boundaries. So, in this sense, MMS can be reorganized as the boundary element method.

3.3. Boundary conditions

In MMS the boundary conditions are formulated directly. There is no need to express some of notions through the other ones as usually done in the theoretical or other numerical method. MMS operates by all essential parameters in explicit form. So, if say, the left or upper side of element is completely free, we put that all force parameters are zeroth:

$$Z_4 = Z_5 = Z_6 = 0 \text{ or } Z_{22} = Z_{23} = Z_{24} = 0. \tag{15a}$$

If, for example, the lower side or right side is clamped, we have:

$$Z_7 = Z_8 = Z_9 = 0 \text{ or } Z_{13} = Z_{14} = Z_{15} = 0. \tag{15b}$$

If, the right side is simply supported, then:

$$Z_{13} = Z_{15} = Z_{16} = 0 . \quad (15c)$$

Which means that displacement, twisting angle and bending moment are equal to zero.

4. Examples of calculations

In this paper we will show only few examples for static isotropic rectangular plates with straight borders parallel to Cartesian coordinates, their length is designated as l_x and height is l_y, where for convenience take that $l_x \le l_y$. Other capabilities, with respect to type of loading (dynamic, harmonical), form of plate (curvilinear boundaries), variable loading and properties within one element – will be analyzed in subsequent investigations. Here we demonstrate the principal advantages, simplicity, consistence, and accuracy of the method on geometries with available exact solutions.

4.1. Rectangular SSSS plate at uniform loading

It is the simplest geometry, Fig. 3, which allow easy to get exact theoretical (Navier) solution. Among other, the theoretical results are given in book of Timoshenko [3]. First of all, demonstrate the consistency of results. Apply the different meshing and investigate the convergence and accuracy of results for particular plate with ratio of height l_y to length l_x equal to 1.5. The results of calculation for different dimensionless parameters ($\boldsymbol{w_x} = \bar{\boldsymbol{w}}_{\boldsymbol{x}} \times \frac{\boldsymbol{q} l_x^{\,4}}{\boldsymbol{D}}$; $\boldsymbol{M_x} = \bar{\boldsymbol{M}}_{\boldsymbol{x}} \times \boldsymbol{q} l_x^{\,2}$; $\boldsymbol{M_y} = \bar{\boldsymbol{M}}_{\boldsymbol{y}} \times \boldsymbol{q} l_x^{\,2}$; $\boldsymbol{Q_x} = \bar{\boldsymbol{Q}}_{\boldsymbol{x}} \times \boldsymbol{q} l_x$; $\boldsymbol{Q_y} = \bar{\boldsymbol{Q}}_{\boldsymbol{y}} \times \boldsymbol{q} l_x$) are given in Table 1.

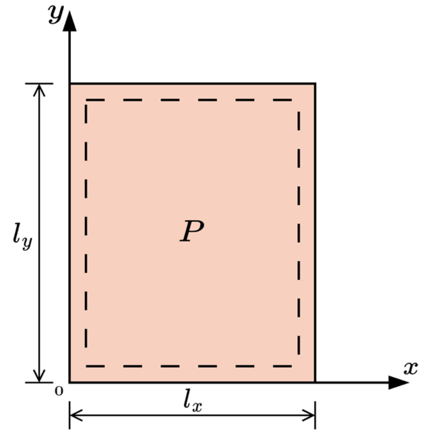

Fig. 3. SSSS plate at uniform loading

It can be stated, that results are consistent, tend to the correct value for finer meshing and demonstrate the similar accuracy for both the geometrical and force parameters; while usually in theoretical or FEM analysis the geometrical parameters are more accurate. So, even 3×3 meshing gives satisfactory results (withing 5% of accuracy). Note that for this meshing, practically, the task can be reduced only to 3×4=12 sides on boundaries with 6 unknowns each, so the task can be reduced to solution of problem with 12×6=72 unknowns. In similar way, state that 7×7 meshing ($\approx 0.1\%$ of accuracy) can be reduced to solution of task with 7×4×6=168 unknowns. And 21×21 meshing can be reduced to calculations with 21×4×6=504 unknowns. Note, that in this example we did not apply the optimization process and solve the problem with maximal number of unknowns, which is equal to $N \times M \times 24$.

Now investigate the accuracy for the same task with different ratio of the plate height to its length. The meshing is fixed and taken as $N = M = 21$. Results of calculations are given in Table 2. The results confirm the above conclusion as to method accuracy.

Table 1. Comparison of results for uniformly loaded rectangular SSSS plate depending on the number of elements ($\nu = 0.3, \frac{l_y}{l_x} = 1.5$)

$N \times N$	$\bar{w}_x\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{M}_x\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{M}_y\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{Q}_x\left(0,\frac{l_y}{2}\right)$	$\bar{Q}_y\left(\frac{l_x}{2},0\right)$
1x1	0.009558	0.098049	0.060641	0.392195	0.161708
3x3	0.008193	0.085391	0.051558	0.431042	0.341498
5x5	0.007889	0.082627	0.050570	0.423888	0.354777
7x7	0.007808	0.081907	0.050228	0.424150	0.359175
11x11	0.007758	0.081462	0.050002	0.423901	0.361967
15x15	0.007742	0.081322	0.049929	0.423844	0.362902
21x21	0.007733	0.081243	0.049887	0.423813	0.363442
Exact [3]	**0.00772**	**0.0812**	**0.0498**	**0.424**	**0.363**

Table 2. Comparison of results for uniformly loaded rectangular SSSS plate ($\nu = 0.3$)

l_y / l_x	Method	$\bar{w}_x\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{M}_x\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{M}_y\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{Q}_x\left(0,\frac{l_y}{2}\right)$	$\bar{Q}_y\left(\frac{l_x}{2},0\right)$
1	Present (19x19)	0.004069	0.047950	0.047950	0.337492	0.337492
	Exact [3]	0.00406	0.0479	0.0479	0.338	0.338
1.1	Present (19x19)	0.004877	0.055559	0.049382	0.359783	0.346175
	Exact [3]	0.00485	0.0554	0.0493	0.36	0.347
1.3	Present (19x19)	0.006402	0.069476	0.050399	0.396344	0.357347
	Exact [3]	0.00638	0.0694	0.0503	0.397	0.357
1.5	Present (19x19)	0.007735	0.081261	0.049897	0.423820	0.363317
	Exact [3]	0.00772	0.0812	0.0498	0.424	0.363
3	Present (19x19)	0.012243	0.118931	0.040609	0.492755	0.367826
	Exact [3]	0.01223	0.1189	0.0406	0.493	0.372

4.2. Rectangular CCCC plate at uniform loading, Fig. 4

The results of calculation are presented in Table 2. The very good accuracy is attained for this relatively simple example for both displacement and bending moments.

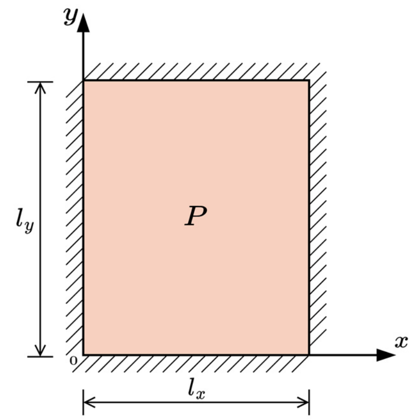

Fig. 4. The scheme of CCCC plate

Let's make the problem a little more difficult, which usually is not investigated in numerical methods, i.e., consider the influence of the ratio between the elements dimensions.

4.3. A very long SSSS plate, Fig. 5

Take that the height of plate exceeds the length by 1000 times, and length is equal to 1. Mesh the plate by 7 columns of equal width (1/7). As to horizontal meshing we draw 5 equal rows of height of 1/5 near upper edge of plate, and the similar 5 rows at the lower edge. So, the height of the central strips is equal to 998, while the length of each is equal to 1/7. It means, that ratio of dimensions for central elements is equal to 998×7=6986 – it is enormous value, and our method copes with it without any problems. The results of calculations are given in Table 3, where calculated moments and displacement actually coincide with those for the SS beam. This reveals a great advantage of our method.

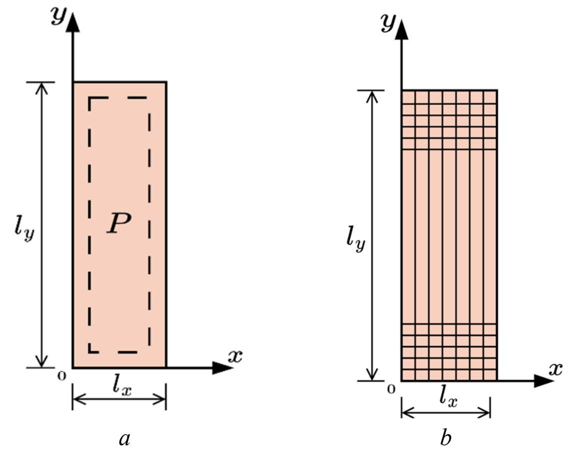

Fig. 5. A very long SSSS plate with $\frac{l_y}{l_x} = 1000$:

a) the scheme of plate; *b*) the scheme of meshing

4.4. Mindlin SFSF plate

There is an exact theoretical solution for it as for Levy type plate [19]. Here in equations (3a) and (3c) we consider the shear deformation of each section. The results of calculation and comparison with exact solution [19] for the square plate are given in Table 5. Evidently the correspondence is very satisfactory and testify about the efficiency of method. No any problem occurs with the locking phenomena at $h / l_x \to 0$. All our results are given for 19×19 meshing.

Table 3. Comparison of results for uniformly loaded rectangular CCCC plate $(\nu = 0.3)$

l_y / l_x	Method	$\bar{w}_x\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{M}_x\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{M}_y\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{M}_x\left(l_x,\frac{l_y}{2}\right)$	$\bar{M}_y\left(\frac{l_x}{2},l_y\right)$
1	Present (19x19)	0.00127010	–0.022962	–0.022962	0.051327	0.051327
	Timoshenko [3]	0.00126	0.0231	0.0231	–0.0513	–0.0513
1,2	Present (19x19)	0.00173121	–0.030048	–0.022888	0.063941	0.055330
	Timoshenko [3]	0.00172	0.0299	0.0228	–0.0639	–0.0554
2	Present (19x19)	0.00253912	–0.041213	–0.015780	0.082931	0.056549
	Timoshenko [3]	0.00254	0.0412	0.0158	–0.0829	–0.0571

Table 4. Comparison of results for uniformly loaded rectangular SSSS plate with extreme sides ratio $(\nu=0.3)$

l_y / l_x	Method	$\bar{w}_x\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{M}_x\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{M}_y\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{Q}_x\left(0,\frac{l_y}{2}\right)$	$\bar{Q}_y\left(\frac{l_x}{2},0\right)$
1000	Present	0.013041	0.125000	0.037229	0.500000	0.337707
∞	Timoshenko [3]	0.01302	0.1250	0.0375	0.500	0.372

Table 5. Comparison of results for uniformly loaded square SFSF Mindlin Plate $(\nu=0.3)$

h / l_x	Method	$\bar{w}_x\left(\frac{l_x}{2},\frac{l_y}{2}\right)$	$\bar{w}_x\left(\frac{l_x}{2},l_y\right)$
0.10	Present	0.013460	0.015541
	exact [19]	0.01346	0.01560
0.15	Present	0.013908	0.016048
	exact [19]	0.01391	0.01616
0.20	Present	0.014535	0.016729
	exact [19]	0.01454	0.01690
0.25	Present	0.015341	0.017581
	exact [19]	0.01536	0.01781
0.30	Present	0.016324	0.018604
	exact [19]	0.01633	0.01889

Conclusion

Principally new version of FEM for plate analysis, namely, Method of Matched Sections, is proposed. In contrast to conventional FEM, where the elements are matched at nodes, in this method the elements are matched by neighboring sides. In fact, the element consists of two almost independent beams – horizontal and vertical ones, each having 6 degrees of freedom as the plane beam – 4 degrees for flexural beam and 2 degrees for the twisting rod. These two beams contain 3 auxiliary constants, which are determined from conditions of geometrical continuity between them – equality of displacement and two angles of rotation at the middle of the plate.

1. The general methodology of numbering the unknowns in the whole system, compiling the equations and algorithm of their solution is proposed.

2. The equations are numbered and compiled in the way convenient for application of transfer matrix method, which is able to drastically decrease the number of equations, and actually reduce the number of unknowns as in boundary element method.

3. Calculations for simple plates show the good efficiency, consistency and accuracy of the method. Furthermore, the plate elements are organically match with beam elements; the ratio of width to length of any element can be absolutely arbitrary.

4. The Mindlin plate analysis shows the very good consistency of the method, which is absolutely free of the locking behavior. The comparison of our results with exact solution for Levy type plate demonstrates the good accuracy of our method for the Mindlin plate.

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Застосування методу узгоджених січень для прямокутних тонкостінних пластин та пластин Міндліна

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Анотація. *У статті розроблено принципово новий варіант методу скінченних елементів у застосуванні до задачі пластин. Він відрізняється від класичного методу скінченних елементів принаймні трьома пунктами. По-перше, він використовує сильну диференціальну постановку, а не слабку, і припускає наближений аналітичний розв'язок усіх диференціальних рівнянь. По-друге, він явно використовує всі геометричні та фізичні параметри в процедурі розв'язання, а не деякі вибрані, наприклад, переміщення та кути, як це зазвичай робиться у формулюванні МСЕ. По-третє, спряження між сусідніми елементами відбувається вздовж перерізів, а не у вершинах багатокутників. Ці умови вимагають неперервності переміщень, кутів, моментів і сил. Кожна сторона прямокутних елементів характеризується 6 основними параметрами, таким чином, в цілому для кожного прямокутного елемента існує 24 параметри. Параметри правої та верхньої граней вважаються вихідними, а з параметрами нижньої та лівої граней вони по'вязані матричними рівняннями, що дозволяє застосувати метод початкових параметрів для складання результуючої системи рівнянь для всієї пластини. Чисельні приклади для пластини Міндліна показують високу ефективність і точність методу.*

Ключові слова: *Метод узгоджених січень, метод початкових параметрів, прямокутна пластина, пластина Міндліна, граничні умови, спряження.*