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Effective decoupling method for derivation of eigenfunctions for closed cylindrical shell

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Abstract. *By expansion into Fourier series along the circumferential coordinate, the problem for elastic thin-walled closed cylindrical shell is reduced to 8th order differential equation with respect to axial coordinate. In spite that the general structure of eigenvalues for this equation was known starting from 60-s of last century, they were derived only to some simplified versions of the shell theory. So, the main goal of paper consists in development of the general procedure for determination of the eigenvalues. The idea is based on that the theory of shell is actually formed by two much simple problems: the plane task of elasticity and the plate problem, each of them is reduced to much easily treated biquadratic equation. So, we start from either of two problems (main problem) while not taking into account the impact from another (auxiliary) problem. After computing its eigenfunctions, we gradually introduce the influence of auxiliary problem by presenting its functions as linear combination of functions for main problem. The results of calculation show the perfect accuracy of the method for any desired number of significant digits in eigenvalues. The comparison with known results for concentrated radial force shows the perfect ability to solve any boundary problem with any desirable accuracy.*

Keywords: *decoupling, coupled problem, closed cylindrical shell, eigenfunction, iterative procedure, main homogeneous equation, auxiliary particular solution, concentrated force.*

Introduction

The problem of a cylindrical thin-walled shell under concentrated radial load can be considered as a coupling of two smaller and simpler problems: the plane (membrane) problem and a plate one. Indeed, due to the shell's non-zero curvature, the radial force from the plate problem gives a projection in a circumferential direction, and, conversely, the circumferential force from the membrane problem gives a projection in a radial direction. Similarly, the equilibrium equations for the membrane problem (in axial and circumferential directions) contain the radial displacement, and the equilibrium equation for the plate problem (in radial direction) contains the axial and circumferential displacements. When the curvature tends to zero, these two problems become independent, uncoupled. The method we propose uses this approach: firstly it considers these two problems as independent ones, but then, for each of two problems, it gradually accounts for the impact of the second "alien" problem.

There exists a large number of different cylindrical thin-walled shell theories, each of them depending on different physical and geometrical assumptions. In any case, the most common approach for solving it is the expansion of all shell parameters in Fourier series in the circumferential direction, e.g. $\cos(n\varphi)$, where n is a circumferential mode. It results in an 8th order ordinary differential equation with respect to the axial coordinate. At least 12 corresponding characteristic equations pertaining to different theories were considered in work [1]. Their roots are compared for few chosen values of n (n = 2, 3, 10), however, these roots are computed using a special procedure based on a solution of biquadric polynomial and generally cannot be expressed in a clear analytical form.

Nonetheless, for some simplified theories these roots can be presented analytically, corresponding expressions are given in work [2] for Donnel's simplified equation. In work [3] authors obtained the simple formulae for the roots of simplified Flugge's equation. In [4], Morley introduced another simplified equation with a simple solution for eigenfunctions. As for more recent researches, the work [5]

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is worth mentioning, where the differential equations were taken from [6].

These findings have the following peculiarities:

1. All characteristic eight eigenfunctions can be presented in a general form:

$$F_{1,2,3,4}(x)=e^{\mp a_n x}\cdot\begin{cases}\cos(b_n x)\\ \sin(b_n x)\end{cases} \tag{1a}$$

$$\Phi_{1,2,3,4}(x)=e^{\mp c_n x}\cdot\begin{cases}\cos(d_n x)\\ \sin(d_n x)\end{cases} \tag{1b}$$

2. For small values of n ($n \ge 2$), the roots of almost all shell characteristic equations are very close and can be separated into two different groups – short-wave group (1a) and long-wave group (1b) [7].

The long-wave (or simply long) solution got more importance in practice. It has obtained the independent significance in literature, despite the fact that it is only a half of the general solution. It was introduced by Vlasov in 1949 as a semi-membrane theory of shells. He introduced two hypotheses, which lead to minor inconsistency: 1) the circumferential strain is zero, and 2) the shear strain is zero. This contradicts to the understanding that the short solution is equally important: for instance, the problem of mitred bend under the inner pressure [8], [9], since "additional stress due to the presence of the miter is very localized" [10]. Nevertheless, Vlasov's theory became very popular in literature. Its application was used in multiple works, such as calculation of end effects in toroidal shells [11], concentrated force [12], loss of bending instability [13], general buckling behaviour [14], vibration analysis of intermediate and short cylindrical shells [15], generalised beam behaviour [16].

Vlasov-like solution can be obtained using a different premise. According to Goldenveizer [18], two main hypotheses can be substituted by only one: the solutions must change more quickly in circumferential direction than in axial one:

$$d^2\Omega/dx^2 \ll d^2\Omega/(Rd\varphi)^2 , \tag{1c}$$

where Ω is any parameter of the shell problem. This idea was fundamental to present authors. By analogy we assumed that, if the long 4th order solution satisfies the aforementioned requirement, then the supplemental short 4th order solution must exist, which must satisfy the opposite to this requirement:

$$d^2\Omega/dx^2 \gg d^2\Omega/(Rd\varphi)^2 \tag{1d}$$

Hence both long and short solutions have equal importance. This idea was developed in our works [19], [20], [20] and eventually polished in [21], where it was accounted that both long and short solutions produce all 8 components (shell parameters) of the complete solution.

In this work we suggest the iterative decoupling procedure for finding the roots of the characteristic equation of coupled problems. We start from simple roots of any out of 2 uncoupled problems (membrane problem and plate one) and slowly refine them by taking into account the influence of another, "alien" problem. It can be shown that the long solution corresponds to the membrane-generated problem, and the short solution – to the plate-generated problem.

Governing equations

Consider an infinite cylindrical shell. Fig. 1 depicts all coordinates (axial x, circumferential φ, radial r) and their positive directions; displacements in corresponding directions (axial u, circumferential v, radial w); membrane forces N_x , N_φ , shear force L, transverse forces Q_x , Q_φ ; bending moments M_x , M_φ , $M_{x\varphi}$.

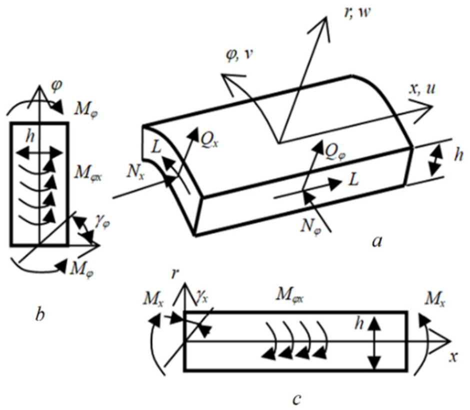

Fig. 1. Directions of geometrical and force parameters: (*a*) 3D view of the element; (*b*) plane (φ, r); (*c*) plane (x, r)

Form the commonly used equilibrium equations:

$$\frac{\partial N_x}{\partial x}+\frac{1}{R}\frac{\partial L}{\partial\varphi}=0,\ \frac{1}{R}\frac{\partial N_\varphi}{\partial\varphi}+\frac{\partial L}{\partial x}+\frac{Q_\varphi}{R}=0,$$

$$\frac{\partial Q_x}{\partial x}+\frac{1}{R}\frac{Q_\varphi}{\partial\varphi}-\frac{N_\varphi}{R}=0,$$

$$Q_x=\frac{M_x}{\partial x}+\frac{1}{R}\frac{\partial M_{x\varphi}}{\partial\varphi},\ Q_\varphi=\frac{1}{R}\frac{\partial M_\varphi}{\partial\varphi}+\frac{\partial M_{x\varphi}}{\partial x} \tag{2a}$$

The above inner forces are related to strains (membrane ε, γ and bending χ) by physical equations:

$$N_x=-H(\varepsilon_x+\mu\varepsilon_\varphi),\ N_\varphi=-H(\varepsilon_\varphi+\mu\varepsilon_x),\ L=-G\varepsilon_{x\varphi}h$$

$$M_x=-H\delta(\chi_x+\mu\chi_\varphi),\ M_\varphi=-H\delta(\chi_\varphi+\mu\chi_x),$$

$$M_{x\varphi}=\frac{-H\delta}{2}(1-\mu)\chi_{x\varphi} \tag{2b}$$

Here the common designations are used:

$H = Eh/\left(1-\mu^2\right)$, $G = E/\left(2(1+\mu)\right)$, $\delta = h^2/12$, μ is Poisson's ratio.

The third set of equations is the geometrical one, it relates the displacements with strains. As for bending strains χ, it is accounted for the fact that the membrane strains ε depend on variable radius and give contributions to the bending strains too [12]. Eventually we will use the following modified expressions for them:

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_\varphi = \frac{1}{R}\frac{\partial v}{\partial \varphi} + \frac{w}{R},\ \varepsilon_{x\varphi} = \frac{1}{R}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial x}$$

$$\gamma_x = \frac{-\partial w}{\partial x},\ \gamma_\varphi = \frac{-1}{R}\frac{\partial w}{\partial \varphi} + \frac{v}{R}$$

$$\chi_x = \frac{-\partial^2 w}{\partial x^2},\ \chi_\varphi = \frac{-\partial^2 w}{R^2 \partial \varphi^2} - \frac{w}{R^2},$$

$$\chi_{x\varphi} = \frac{-2\partial^2 w}{R\partial\varphi\partial x} - \frac{1}{R^2}\frac{\partial u}{\partial \kappa} + \frac{1}{R}\frac{\partial v}{\partial x} \quad (2c)$$

The formula for $\chi_{x\varphi}$ can be simplified more (the error due to the following approximation is pretty small and does not exceed 1% [21]). We will use the next expression:

$$\chi_{x\varphi} = \frac{-2\partial^2 w}{R\partial\varphi\partial x} \quad (2d)$$

Expand unknown functions (shell parameters) in Fourier series along the circumferential coordinate. They can be presented as follows:

$$\left\{w(x,\varphi); u(x,\varphi)\right\} = \left\{w_n(x); u_n(x)\right\}\cdot\cos(n\varphi);$$

$$v(x,\varphi) = v_n(x)\cdot\sin(n\varphi)$$

$$\left\{N_x(x,\varphi); N_\varphi(x,\varphi); Q_x(x,\varphi); M_x(x,\varphi); M_\varphi(x,\varphi)\right\} =$$
$$= \left\{N_{x,n}(x); N_{\varphi,n}(x); Q_{x,n}(x); M_{x,n}(x); M_{\varphi,n}(x)\right\}\cdot\cos(n\varphi)$$

$$\left\{L(x,\varphi); Q_\varphi(x,\varphi); M_{x\varphi}(x,\varphi)\right\} =$$
$$= \left\{L_n(x); Q_{\varphi,n}(x); M_{x\varphi,n}(x)\right\}\cdot sin(n\varphi) \quad (2e)$$

In subsequent analysis we will mostly omit the lower index "n", implying that all these parameters depend on n. One may notice that this notation for Fourier series expansion factors coincides with basic notation of shell parameters, but it will hardly ever lead to any confusion. If we want to use the basic notation for any function, we will explicitly state that this function depends on two coordinates, x and φ. Otherwise it will be implied that the function depends solely on x, so this function is the expansion factor.

Description of iterative decoupling procedure

As mentioned before, the fundamental problem of a circular cylindrical shell is a coupling of two interdependent problems: the plane (membrane) problem and a plate one. Due to the shell's non-zero curvature, the radial force from the plate problem gives a projection in a circumferential direction, and, conversely, the circumferential force from the membrane problem gives a projection in a radial direction. The equilibrium equations in axial and circumferential equations correspond to the membrane problem, and the equilibrium equation in radial direction corresponds to the plate problem.

In order to decouple our two coupled problems, first consider only one of them with no impact from the other, as though it is fully independent. It means that in corresponding governing equation all terms containing functions from another "alien" problem should be absent. Find eigenvalues and eigenfunctions of this chosen problem (it will be referred as a "main" one), and obtain the expressions for unknown functions which characterise main problem. Now it is time to gradually account for the impact of another, "alien" problem. By slowly doing this, we gradually move from the solution of independent uncoupled problem to the solution of fully interconnected coupled problems, which we had in the beginning. That is the essence of our method. First, use the governing equation of the second ("alien", "auxiliary") problem and find its partial solution. It is possible, because all eigenfunctions and other unknown functions from the main problem are already known. Thus all the "alien" functions, which characterise this auxiliary problem, become known too. The last step is actually introducing the impact of the "alien" problem: return to the governing equation of the main problem, but now do not neglect terms containing "alien" functions. Instead express them as linear combinations of terms from the "main" problem. It is possible too, since all these terms and functions consist of the same eigenfunctions. Such step allows to 1) turn non-homogeneous main equation into a homogeneous one, and 2) to take into account some impact from the "alien" problem. Now repeat all the steps above: keep finding eigenfunctions and keep introducing the influence of the "alien" problem until convergence is reached, until eigenfunctions stop changing. When finished, switch between main and auxiliary problems: let the auxiliary problem become a main one, and the main problem – an auxiliary one.

Let us explain the decoupling procedure more clearly on a specific example of an infinite cylindrical shell under radial load. The equilibrium equations in axial, circumferential, and radial direction respectively are following (the "alien" terms were moved to the right side of the equations):

$$\frac{d^2u}{dx^2} + \frac{n(1+\mu)}{2R}\frac{dv}{dx} - \frac{n^2(1-\mu)}{2R^2}u = \frac{-\mu}{R}\frac{dw}{dx} = RS1 \quad (3a)$$

$$-\left(\frac{1-\mu}{2}\right)\frac{d^2v}{dx^2} + \frac{n(1+\mu)}{2R}\frac{du}{dx} + \frac{n^2}{R^2}v =$$

$$=\frac{\delta n}{R^2}\frac{d^2w}{dx^2}+\left(-\frac{n}{R^2}\right)\left(1+\frac{\delta\left(n^2-1\right)}{R^2}\right)w=RS2 \quad (3b)$$

$$\delta\frac{d^4w}{dx^4}+\left(\frac{-\delta\left(2n^2-\mu\right)}{R^2}\right)\frac{d^2w}{dx^2}+\left(\frac{\delta n^2\left(n^2-1\right)}{R^4}+\frac{1}{R^2}\right)w=$$

$$=\left(-\frac{\mu}{R}\right)\frac{du}{dx}+\left(-\frac{n}{R^2}\right)v=RS3. \quad (3c)$$

Recall that the equations (3a)–(3b) constitute the membrane problem, and the equation (3c) makes up the plate problem. To achieve more compact form, denote coefficients next to unknown functions and their derivatives in the left side as α_{v1}, α_{u0} for the first equation, β_{v2}, β_{u1}, β_{v0} for the second equation, γ_{w4}, γ_{w2}, γ_{w0} for the third equation. The decoupling procedure is ready to start. To find the first two eigenfunctions out of four, consider the membrane problem as a main one, and the plate problem as an auxiliary one.

Step 1 – main problem. Suppose that in the previous iteration we found the expression of the right-side "alien" terms containing $w(x)$ via left-side terms containing the "native" functions $u(x)$, $v(x)$. In other words, suppose we know such real numbers C_u, C_v, D_u, D_v that:

$$RS1=\frac{-\mu}{R}\frac{dw}{dx}=C_u u+C_v\frac{dv}{dx} \quad (4a)$$

$$RS2=\frac{\delta n}{R^2}\frac{d^2w}{dx^2}+\left(-\frac{n}{R^2}\right)\left(1+\frac{\delta\left(n^2-1\right)}{R^2}\right)w=D_u\frac{du}{dx}+D_v v \quad (4b)$$

Recall that such presentation is possible because all unknown functions $w(x)$, $u(x)$, $v(x)$ with their derivatives are the linear combinations of the same eigenfunctions – products of an exponent and a trigonometric function. Also keep in mind that in the first iteration all coefficients C_u, C_v, D_u, D_v equal to zero (the "alien" impact is absent). Now, when both sides of membrane equations contain the same terms, we can reduce the like terms and simplify the equations:

$$\frac{d^2u}{dx^2}+\left(\alpha_{v1}-C_v\right)\frac{dv}{dx}+\left(\alpha_{u0}-C_u\right)u=0 \quad (4c)$$

$$\beta_{v2}\frac{d^2v}{dx^2}+\left(\beta_{u1}-D_u\right)\frac{du}{dx}+\left(\beta_{v0}-D_v\right)v=0 \quad (4d)$$

Combine these two differential equations into one, solve it and find its eigenvalues and eigenfunctions:

$$\lambda_{m,1,2}=-c\pm di,\ c>0,\ d>0 \quad (4e)$$

$$\Phi_1(x)=e^{-cx}\cos(dx),\ \Phi_2(x)=e^{-cx}\sin(dx) \quad (4f)$$

The coefficient *c* must be negative, because we consider an infinite shell – eigenfunctions must decay to zero when *x* approaches infinity. Also one important remark must be made regarding the value of the coefficient *d*: in some cases, *d* equals to zero, which makes eigenvalues real and changes the form of eigenfunctions. For instance, it happens in the very first iteration: the 2 roots of the characteristic equation equal to $-n/R$. In such cases, forcefully introduce a small complex perturbation: for example, $d=0.001n/R$.

Now, when the eigenfunctions are found, we can easily obtain "native" functions $u(x)$ and $v(x)$. In general, they equal to linear combinations of the eigenfunctions, and each eigenfunction must be multiplied by a coefficient computed from boundary conditions. However, our goal is to find eigenfunctions, and boundary conditions can be applied later. Therefore we can choose these coefficients arbitrarily: $u(x)=1\cdot\Phi_1(x)+0\cdot\Phi_2(x)$, $v(x)=A_{v,1}\cdot\Phi_1(x)+$ $+A_{v,2}\cdot\Phi_2(x)$, where the coefficients, $A_{v,1}$, $A_{v,2}$ can be found from (4c).

Step 2 – auxiliary problem. On the first step, we found the "native" functions for the main problem – displacements $u(x)$, $v(x)$. Now let us find its "alien" function $w(x)=A_{w,1}\Phi_1(x)+A_{w,2}\Phi_2(x)$ using the auxiliary problem, accounting for the new eigenvalues and eigenfunctions. In order to do this, substitute found functions $u(x)$, $v(x)$ and the expression for unknown function $w(x)$ into the plate equation (3c). Find coefficients $A_{w,1}$, $A_{w,2}$ – and thus, "alien" displacement $w(x)$ becomes known. To close the iteration loop and start the next iteration, express the "alien" terms of the main equation via its "native" terms – in other words, find such coefficients C_u, C_v, D_u, D_v that expressions (4a) and (4b) hold true. Now step 2 and the whole iteration is ended, next iterations may start, where we refine all functions $w(x)$, $u(x)$, $v(x)$ and eigenfunctions $\Phi_1(x)$, $\Phi_2(x)$ until the convergence is reached.

When finished, switch between main and auxiliary problems: let the plate problem, which used to be an auxiliary problem, become a main one, and the membrane problem – an auxiliary one. The procedure of solving is similar to what we have described: find eigenvalues and eigenfunctions of the main problem, find its "native" functions, find its "alien" functions using the auxiliary problem, and close the iteration loop by expressing "alien" terms via "native" ones. Thus the resting two out of four eigenfunctions will be found, and boundary conditions may finally be applied.

Results

First, we applied the decoupling procedure for the problem of cylindrical shell and computed its four eigenfunctions. Then we considered the action of concentrated radial force and applied boundary conditions:

$$u\big|_{x=0}=0\,,\ \gamma_x\big|_{x=0}=0\,,$$

$$Q_x\big|_{x=0}=\frac{P}{2},P\left(x,\varphi\right)=2\delta\left(x\right)\cos\left(nx\right),\ L\big|_{x=0}=0 \quad (5a)$$

Afterwards we computed values of most practical interest: radial displacement w and bending moment M_x at $x=0$. Specifically, as in our work [21], we found their dimensionless values $\tilde{w}$ and $\tilde{M}_x$, where:

$$\tilde{w}=\frac{w}{C_w},\ \tilde{M}_x=\frac{M_x}{C_M},\ C_w=\frac{\left(1-\mu^2\right)^{\frac{3}{4}}}{E}\left(\frac{R}{h}\right)^{\frac{5}{2}},$$

$$C_M=\frac{-\sqrt{Rh}}{2\sqrt[4]{3\left(1-\mu^2\right)}} \quad (5b)$$

At last, we compared obtained results with similar values calculated in [21] by exact Navier method in order to find the accuracy of our method (Tab. 1).

As we see, the results are in very remarkable agreement, while our present results can be considered as the most accurate ones. Also, as was noted in [21], the application of Navier method is very restricted, whereas the present method has much broader applications, especially if we will apply the ascending eigenfunctions too, which can be derived in the same way as considered here decaying eigenfunctions.

By collecting more data points, it is possible to find an approximate analytical expression for dimensionless radial displacement $\tilde{w}(x)$ and bending moment $\tilde{M}_x(x)$ at $x=0$. Tables 2 and 3 contain these computed values for various *n* and *R*/*h*. Let us focus on the most common case,

Table 1. Dimensionless values of radial displacement $\tilde{w}$ and bending moment $\tilde{M}_x$ at $x=0$ for decoupling procedure and exact Navier method, $R/h=40$.

R/*h* = 40	Navier method		Decoupling procedure	
n	$\tilde{w}(0)$	$\tilde{M}_x(0)$	$\tilde{w}(0)$	$\tilde{M}_x(0)$
2	1.81360	1.41821	1.813599	1.41829
4	0.35259	1.49125	0.352586	1.49133
6	0.14242	1.41783	0.142416	1.41791
8	0.067831	1.21572	0.0677314	1.21575
10	0.036151	1.02004	0.0361505	1.02011
20	0.0046372	0.52752	0.00463717	0.52755
40	0.00057952	0.26421	0.000579521	0.26423
60	0.00017165	0.17614	0.000171649	0.17615

Table 2. Dimensionless values of radial displacement $\tilde{w}(0)$ for various *n* and R/h

R/*h*	*n* = 2	*n* = 3	*n* = 4	*n* = 5	*n* = 6	*n* = 10	*n* = 20	*n* = 40
20	1.85741	0.67359	0.34222	0.19366	0.11740	0.02631	0.00329	0.00041
30	1.82996	0.66579	0.35283	0.21110	0.13371	0.03183	0.00402	0.00050
40	1.81360	0.65636	0.35259	0.21747	0.14242	0.03615	0.00464	0.00058
60	1.79556	0.64293	0.34659	0.21892	0.14889	0.04241	0.00566	0.00071
80	1.78593	0.63471	0.34104	0.21665	0.14970	0.04647	0.00651	0.00082
100	1.77996	0.62931	0.33685	0.21406	0.14887	0.04907	0.00724	0.00092
150	1.77180	0.62159	0.33027	0.20907	0.14579	0.05199	0.00870	0.00112
1000	1.75738	0.60715	0.31661	0.19656	0.13477	0.04877	0.01300	0.00265
5000	1.75529	0.60499	0.31445	0.19442	0.13265	0.04682	0.01195	0.00321

Table 3. Dimensionless values of bending moment $\widetilde{M}_x(0)$ for various n and R/h

R/h	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 10$	$n = 20$	$n = 40$
20	1.47496	1.50764	1.45461	1.32187	1.17010	0.74241	0.37375	0.18686
30	1.43896	1.48171	1.49566	1.43922	1.33193	0.89815	0.45738	0.22885
40	1.41829	1.45523	1.49133	1.48096	1.41791	1.02011	0.52755	0.26423
60	1.39592	1.41899	1.46141	1.48804	1.48080	1.19652	0.64407	0.32355
80	1.38412	1.39721	1.43515	1.47048	1.48747	1.31092	0.74045	0.37350
100	1.37685	1.38300	1.41557	1.45141	1.47806	1.38402	0.82326	0.41743
150	1.36694	1.36277	1.38506	1.41517	1.44553	1.46605	0.98967	0.51061
1000	1.34955	1.32521	1.32223	1.32539	1.33162	1.37243	1.47834	1.21030
5000	1.34704	1.31961	1.31235	1.31010	1.30988	1.31695	1.35842	1.46474

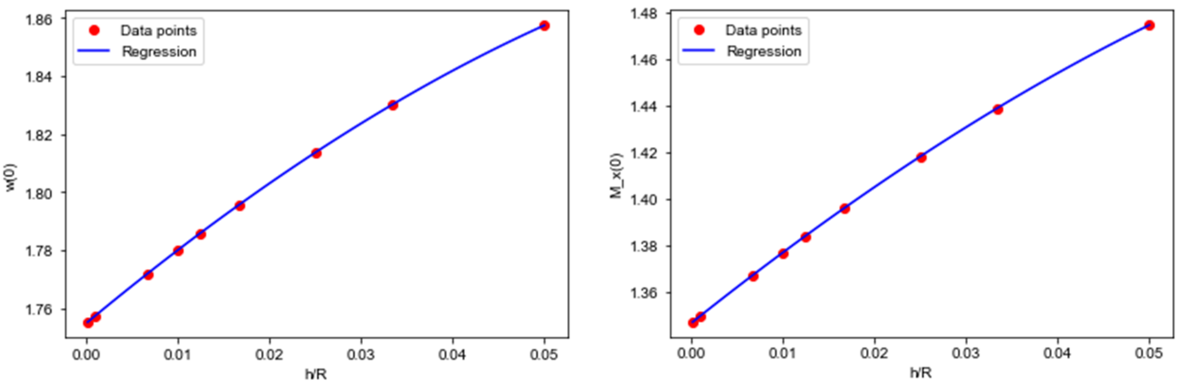

Fig. 2, 3. Results of linear regression for dimensionless radial displacement (left) and bending moment (right) at $x = 0$

$n = 2$. Perform linear regression and find a polynomial function $\tilde{w}\left(\frac{h}{R}\right)\Big|_{x=0} = a_0 + a_1\frac{h}{R} + a_2\left(\frac{h}{R}\right)^2 + a_3\left(\frac{h}{R}\right)^3$ that is the best fit for given values of displacement (the same applies to bending moment too).

By doing this, obtain the following:

$$\tilde{w}\left(\frac{h}{R}\right)\Big|_{x=0} = 1.7548 + 2.6271\frac{h}{R} - \\ -10.4259\left(\frac{h}{R}\right)^2 - 21.1171\left(\frac{h}{R}\right)^3 \tag{6a}$$

$$\tilde{M}_x\left(\frac{h}{R}\right)\Big|_{x=0} = 1.3464 + 3.1526\frac{h}{R} - \\ 10.5732\left(\frac{h}{R}\right)^2 - 21.1136\left(\frac{h}{R}\right)^3 \tag{6b}$$

For both cases the coefficient of determination R^2 is higher than 0.9999999, which signifies the very high accuracy of regression. Figs. 2 and 3 demonstrate fitted polynomials and available data points of dimensionless radial displacement and bending moment.

Conclusions

In this work we presented the problem of a cylindrical infinite shell under radial load as a coupling of two problems (membrane and plate), and introduced the iterative decoupling method for solving these coupled problems. The idea of this method consists in gradual accounting for the impact of the “alien” problem on the “main” problem, starting from no impact at all. One of the problems is considered as a main one and is used to compute eigenfunctions. The other problem is considered as an auxiliary one and is used to introduce the impact on the main problem, and therefore to bring obtained solution closer to the solution of fully interconnected coupled problems. The iterative process is terminated when the change of eigenvalues and eigenfunctions becomes negligible.

The results of calculation show the perfect accuracy of the method for any required number of significant digits in eigenvalues. The comparison with results of work [21] shows the perfect ability to solve any boundary problem with any desirable accuracy. Approximate analytical formulae are suggested for dimensionless values of most practical interest – radial displacement $\tilde{w}$ and bending moment $\tilde{M}_x$ at $x = 0$, the coefficient of determination shows very high accuracy of the regression.

Finally, this procedure can be applied not only to the cylindrical shell, but, in general, to any coupled problems provided that it is possible to express the "alien" terms of the main equation via its "native" terms. This allows for further development of theory of thin-walled structures and other related theories too.

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Ефективна процедура роз'єднання для виведення власних функцій замкненої циліндричної оболонки

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Анотація. *Шляхом розкладу в ряд Фур'є за коловою координатою, задача пружної тонкостінної замкненої циліндричної оболонки зводиться до диференціального рівняння 8-го порядку відносно осьової координати. Попри те, що загальна структура власних чисел для цього рівняння була відома ще з 60-х років минулого століття, вони були отримані лише для деяких спрощених версій теорії оболонок. Таким чином, основна ціль статті полягає в розробці загальної процедури для визначення власних чисел. Ідея базується на тому, що теорія оболонок насправді сформована двома значно простішими задачами: плоскою задачею теорії пружності та задачею про пластину, кожна з них зводиться до простого біквадратного рівняння. Метод починається з будь-якої з двох задач (головна задача), не враховуючи вплив іншої (допоміжної) задачі. Після обчислення власних функцій ми поступово вводимо вплив допоміжної задачі шляхом представлення її функцій як лінійних комбінацій функцій головної задачі. Результати обчислень показують чудову точність методу для будь-якого числа значущих цифр у власних числах. Порівняння з відомими результатами для зосередженої радіальної сили демонструє чудову здатність методу розв'язувати будь-які граничні задачі з довільною бажаною точністю.*

Ключові слова: *роз'єднання, зв'язані задачі, замкнена циліндрична оболонка, власні функції, ітеративна процедура, головне однорідне рівняння, допоміжний часткове розв'язок, зосереджена сила.*