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Implicit direct time integration of the heat conduction problem in the Method of Matched Sections

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Abstract: *The paper is devoted to further elaboration of the Method of Matched Sections as a new branch of finite element method in application to the transient 2D temperature problem. The main distinction of MMS from conventional FEM consist in that the conjugation is provided between the adjacent sections rather than in the nodes of the elements. Important feature is that method is based on approximate strong form solution of the governing differential equations called here as the Connection equations. It is assumed that for each small rectangular element the 2D problem can be considered as the combination of two 1D problems – one is x-dependent, and another is y-dependent. Each problem is characterized by two functions – the temperature, T, and heat flux Q . In practical realization for rectangular finite elements the method is reduced to determination of eight unknowns for each element – two unknowns on each side, which are related by the Connection equations, and requirement of the temperature continuity at the center of element. Another salient feature of the paper is an implementation of the original implicit time integration scheme, where the time step became the parameter of shape function within the element, i.e. it determines the behavior of the Connection equations. This method was early proposed by first author for number of 1D problem, and here in first time it is applied for 2D problems. The number of tests for rectangular plate exhibits the remarkable properties of this "embedded" time integration scheme with respect to stability, accuracy, and absence of any restrictions as to increasing of the time step.*

Keywords: *Method of Matched Sections, implicit time integration, time step dependent shape functions, rectangular plate, transient temperature.*

1. Introduction

Transient heat conduction is a common phenomenon in many natural and engineering systems [1]. Its analysis requires the application of the time integration methods. During the past decades, many kinds of numerical and analytical methods have been proposed for analysis of the practical transient heat transfer engineering problem [2]–[4]. These applications require numerous and accurate repeated calculation with the aim of optimization of the shape and technology parameters [5] or restoring the impact of unknown heat source in inverse problems [6], [7].

Analytical methods are capable to grasp the physical insight of the problem, determine the role and significance of each inner or outer parameter of the problem, to formulate the dimensionless combinations of essential parameters which predetermines the solution behavior. The most available and popular is the classical method of separation of variables [8], which can be easily applied to the easily applied to bodies of the canonical form (circular, rectangular). More advanced analytical methods are based on Green's function method [9]–[11], the integral transform method [12], [13]. Nevertheless, the analytical methods are mainly limited to problems with simple geometries and boundary conditions, and mostly used for verification of various numerical approaches [14].

So mostly various numerical methods are adopted for solution of practical task. Among them are the boundary element method [15], [16], meshless method [17] and the finite element method [19] as the most effective tool realized in various commercial software.

On the other hand, the thermal transient solution is often only a prerequisite for performing the thermal stress and deformation analysis of mechanical structures. The mechanical and temperature tasks are handled by the same

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researchers, the similar methods are used for both tasks [20] and presented in the same textbooks [21].

This paper is generalization of our "mechanical" experience for application to transient heat conduction problem. Two main ideas will be described below. First one is related with the Method of Matched Sections as a variant of FEM. The second one is concerned with the time integration in MMS.

1. Classical FEM present the structure as consisting of large number of finite elements where the degrees of freedom (unknowns) are related with some parameters at the nodes. Other parameters of the problem are related with them by the application of the procedure of minimization of the energy functional or by Galerkin minimization procedure. The drawback of classical FEM is that equilibrium is satisfied in a weak sense, i.e. "they are not in equilibrium with the body forces and do not have tractions that equilibrate with the static boundary conditions and are not continuous between elements" [22]. The same is noted by authors of [23], who emphasized that Newton's third law is therefore violated at the boundaries between elements.

The main idea of works [24], [25] is that solution within small 2D rectangular element can be presented as combination of two separate 1D problems, each of them is dependent only from one coordinate x or y. Two separate problems are united by continuity condition in the middle of element. For 2D plate deformation the solutions with respect of any coordinate axis closely resemble that for 1D beam problem, so we titled our method [24] as a "beam-like approach". Here for the temperature task, there is no beam analogy, but the essence of the method remains the same: we attribute two sets of physical unknowns (temperature, T and heat flux, Q) at the middle of each side and consider that these values are functions of only one coordinate axis which is directed normally to the side considered.

2. The second novelty of the paper is related with specific direct time integration procedure. It is known that the standard finite element method is not very effective for the solution of wave propagation problems [21], [26]. For example, for harmonic wave solution the accuracy of solution noticeably deteriorates with increasing wave number [27]. As a remedy for this problem, it is suggested to use in the element interpolation functions the additional degrees of freedom corresponding to very quickly changed within element the harmonical functions [26]. Interesting to note, that enriched element interpolation functions as given in [26] were used in analysis of wave propagation in a rod [28]. Alternative idea was proposed in our works [29], [30] where time step in implicit central difference scheme is used directly in element interpolation schemes. The similar idea (without realization) was expressed by Reddy in the textbook [31]. The integration method was named in 1D problem as the semianalytical one [29]. With application to 2D problems in MMS, as it will be shown below, the value of time step is directly used in approximate analytical solution within the element. So, the procedure of MMS remains the same for static and dynamic cases, while affecting on the "element interpolation functions".

2. Basic ideas of MMS in application to heat conduction.

2.1. Governing equations of MMS

Consider the rectangular element. Consider the heat conduction task as a combination of two independent problems in two perpendicular directions – x and y. To distinguish them introduce the notions $T^x(x)$ and $Q^x(x)$ as functions related to the temperature and heat flux in x direction, and $T^y(y)$ and $Q^y(y)$ as the functions in y direction. Upper indexes x and y show that the respected values are attributed to the sides which are perpendicular to the axes x and y correspondently, Fig1. Write down the governing Fourier heat conduction equations in two perpendicular directions [1]:

$$Q^x(x,t) = -k^x \frac{dT^x(x,t)}{dx}, \tag{1a}$$

$$Q^y(y,t) = -k^y \frac{dT^y(y,t)}{dy}. \tag{1b}$$

Where t – is the time variable and k^x and k^y are heat conduction coefficients and the local coordinates in the element are within the ranges $0 \le x \le a$, and $0 \le y \le b$.

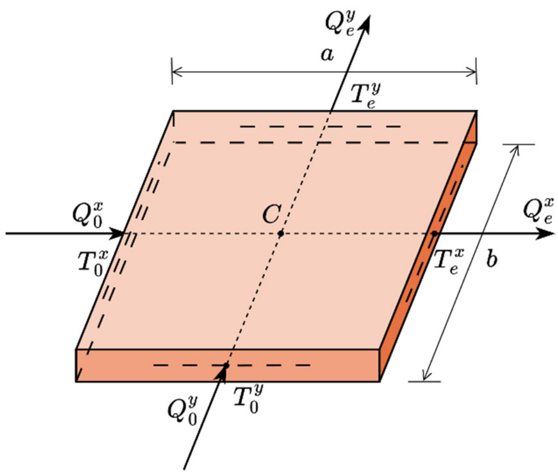

Fig 1. The main parameters in the rectangular element

As the next governing equation consider the law of heat energy conservation. In absence of outer heat sources, we have [1]:

$$\frac{\partial Q(x,y,t)}{\partial x} + \frac{\partial Q(x,y,t)}{\partial y} = -c\rho \frac{\partial T(x,y,t)}{\partial t}, \tag{2a}$$

where c and ρ are heat capacity and density. We have the derivative with respect to time in the right side of equation (2a). Apply the implicit finite difference scheme to the time derivative and account for that heat flux consists of two separate functions of x and y:

$$\frac{dQ_i^x(x)}{dx}+\frac{dQ_i^y(y)}{dy}=-c\rho\frac{T_i(x,y)-T_{i-1}(x,y)}{\Delta}. \quad (2b)$$

Where Δ is the time step used and lower index i designates the i-th time iteration. Equation (2b) cannot be used directly in MMS, because this equation contains both space variables x and y. So, we need to separate this single equation into two ones, each should contain only one space variable. With this goal present the temperature in time i, $T_i(x,y)$, as:

$$T_i(x,y)=T_i^x(x)+T_i^y(y)-T_{c,i}. \quad (2c)$$

Where $T_{c,i}$ is the value of the temperature at the time t_i in the central point of element C, Fig 1, and is related with two "independent" temperatures as:

$$T_{c,i}=T_i^x(x=a/2)=T_i^y(y=b/2). \quad (2d)$$

The equation (2d) is an important continuity equation which "glue" two solutions in the perpendicular directions.

Present the temperature, $T_{i-1}(x,y)$, at previous moment of time $i-1$ as the polynomial expansion:

$$T_{i-1}(x,y)=a_{i-1}^{0,0}+a_{i-1}^{1,0}x+a_{i-1}^{0,1}y+a_{i-1}^{2,0}x^2+a_{i-1}^{0,2}y^2$$
$$=a_{i-1}^{0,0}+\sum_{m=1}^{2}a_{i-1}^{m,0}x^m+\sum_{k=1}^{2}a_{i-1}^{0,k}y^k. \quad (2e)$$

Note, that we the degree of this polynomial expansion should be no less than 1, i.e. it should be capable to grasp the static solution [29]. In given work we take that degree of polynomial expansion is equal to 2. Thus substitute the temperature presentations (2c)–(2e) into right side of (2b), we get:

$$\frac{dQ_i^x(x)}{dx}+\frac{dQ_i^y(y)}{dy}=-c\rho\frac{\left(T_i^x(x)-a_{i-1}^{1,0}x-a_{i-1}^{2,0}x^2\right)+\left(T_i^y(y)-a_{i-1}^{0,1}y-a_{i-1}^{0,2}y^2\right)-\left(T_{c,i}+a_{i-1}^{0,0}\right)}{\Delta}. \quad (3a)$$

Now we are able to separate the governing equation (2f) into two coordinates independent equations:

$$\frac{dQ_i^x(x)}{dx}=-c\rho\frac{\left(T_i^x(x)-a_{i-1}^{1,0}x-a_{i-1}^{2,0}x^2\right)-\left(T_{c,i}+a_{i-1}^{0,0}+A_i\right)}{\Delta} \quad (3b)$$

$$\frac{dQ_i^x(y)}{dy}=-c\rho\frac{\left(T_i^y(y)-a_{i-1}^{0,1}y-a_{i-1}^{0,2}y^2\right)+A_i}{\Delta}. \quad (3c)$$

Where we introduced the auxiliary constant A_i, which account for the possible redistribution of the heat flux within the element between two "independent" directions.

To sum up this subchapter 2.1 note, that equations (1a) and (1b) as well as (3b) and (3c) are two groups of the main governing equations of the transient heat conduction problem by MMS.

2.2. General solution of the main equations by MMS

Substitute equations (1a) and (1b) into (3b) and (3c) correspondently. This gives two 2nd degree differential inhomogeneous equations with respect two independent functions $T_i^x(x)$ and $T_i^y(y)$:

$$\frac{d^2T_i^x(x)}{dx^2}-b_x^2T_i^x(x)=-b_x^2\left(a_{i-1}^{0,0}+a_{i-1}^{1,0}x+a_{i-1}^{2,0}x^2+T_{c,i}+A_i\right) \quad (4a)$$

$$\frac{d^2T_i^y(y)}{dy^2}-b_y^2T_i^y(y)=-b_y^2\left(a_{i-1}^{0,1}y+a_{i-1}^{0,2}y^2-A_i\right). \quad (4b)$$

Where the following designations are introduced:

$$b_x^2=\frac{c\rho}{\Delta k^x}; \qquad b_y^2=\frac{c\rho}{\Delta k^y}. \quad (4c)$$

Obtain their general solution. Start from general solution of the homogeneous part of equation (4a), i.e. get the x general solution of the below equation:

$$\frac{d^2T_i^x(x)}{dx^2}-b_x^2T_i^x(x)=0. \quad (5a)$$

Introduce the generalized Krylov's functions which have the remarkable properties in the point $x=0$:

$$K_1(x)=ch(xb_x), \quad K_2(x)=sh(xb_x)/b_x. \quad (5b)$$

Then the general solution, $\overline{T_i^x}(x)$, is given by the following formula:

$$\overline{T_i^x}(x)=C_1K_1(x)+C_2K_2(x). \quad (5c)$$

Accounting for the initial values of the temperature and the heat flux at the beginning of the element in the horizontal direction, i.e. in point $x=0$ we can write:

$$\overline{T_i^x}(x=0)=T_{i,0}^x, \quad \overline{Q_i^x}(x=0)=-k^x\frac{dT^x(x=0)}{dx}=Q_{i,0}^x. \quad (5d)$$

Where from the general solution of the homogeneous equation is presented as:

$$\overline{T_i^x}(x)=T_{i,0}^xK_1(x)-Q_{i,0}^x\frac{K_2(x)}{k^x}. \quad (5e)$$

After that find the usual particular solution, $\tilde{T}_i^x(x)$, of (4a):

$$\tilde{T}_i^x(x)=\left(a_{i-1}^{0,0}+\frac{2a_{i-1}^{2,0}}{b_x^2}+a_{i-1}^{1,0}x+a_{i-1}^{2,0}x^2+T_{c,i}+A_i\right). \quad (6a)$$

Since the partial solution must satisfy zero initial conditions (in point $x=0$), then find the supplemental (zeroth) particular solution for the temperature:

$$\hat{T}_i^x(x)=\left(a_{i-1}^{0,0}+\frac{2a_{i-1}^{2,0}}{b_x^2}+T_{c,i}+A_i\right)$$
$$\times\left(1-K_1(x)\right)+a_{i-1}^{1,0}\left(x-K_2(x)\right)+a_{i-1}^{2,0}x^2. \quad (6b)$$

The complete solution, $T_i^x(x)$, is the sum of general and supplemental particular solutions, so the expression for it has the form:

$$T_i^x(x)=T_{i,0}^xK_1(x)-Q_{i,0}^x\frac{K_2(x)}{k^x}+\left(a_{i-1}^{0,0}+\frac{2a_{i-1}^{2,0}}{b_x^2}+T_{c,i}+A_i\right)$$
$$\times\left(1-K_1(x)\right)+a_{i-1}^{1,0}\left(x-K_2(x)\right)+a_{i-1}^{2,0}x^2. \quad (6c)$$

Similarly, the complete solution for heat flux, $Q_i^x(x)$, is found from (1a):

$$Q_i^x(x)=-k^xb_x^2T_{i,0}^xK_2(x)+Q_{i,0}^xK_1(x)+k^xb_x^2K_2(x)$$
$$\times\left(a_{i-1}^{0,0}+\frac{2a_{i-1}^{2,0}}{b_x^2}+T_{c,i}+A_i\right)-k^xa_{i-1}^{1,0}\left(1-K_1(x)\right)-2k^xa_{i-1}^{2,0}x. \quad (6d)$$

In similar manner get the complete solution for the temperature and heat distribution in the y direction:

$$T_i^y(y)=T_{i,0}^yK_1(y)-Q_{i,0}^y\frac{K_2(y)}{k^y}+\left(\frac{2a_{i-1}^{0,2}}{b_y^2}-A_i\right)$$
$$\times\left(1-K_1(y)\right)+a_{i-1}^{0,1}\left(y-K_2(y)\right)+a_{i-1}^{0,2}y^2, \quad (7a)$$

$$Q_i^y(y)=-k^yb_y^2T_{i,0}^yK_2(y)+Q_{i,0}^yK_1(y)+k^yb_y^2K_2(y)$$
$$\left(\frac{2a_{i-1}^{0,2}}{b_y^2}-A_i\right)-k^ya_{i-1}^{0,1}\left(1-K_1(y)\right)-2k^ya_{i-1}^{0,2}y^2. \quad (7b)$$

2.3. Algorithm of solution

Equations (6c), (6d) and (7a), (7b) is the main result of the analytical model. Yet they contain the unknown coefficient A_i and the temperature in the center of element $T_{c,i}$, which hinder the direct application of the method. To eliminate them from consideration, apply two conditions (2d) at the center of element. Substituting in them the solutions (6c) and (7a) we get the explicit expressions for these constants:

$$A_i=-\frac{\gamma_yK_1(a/2)+\gamma_x}{2-K_1(a/2)-K_1(b/2)}, \quad (8a)$$

$$T_{c,i}=\frac{\left(\gamma_yK_1(a/2)+\gamma_x\right)\left(1-K_1(b/2)\right)}{2-K_1(a/2)-K_1(b/2)}+\gamma_y. \quad (8b)$$

Where the auxiliary constants γ_x and γ_y are given by the following expressions:

$$\gamma_x=T_{i,0}^xK_1(a/2)-Q_{i,0}^x\frac{K_2(a/2)}{k^x}+\left(a_{i-1}^{0,0}+\frac{2a_{i-1}^{2,0}}{b_x^2}\right)$$
$$\times\left(1-K_1(a/2)\right)+a_{i-1}^{1,0}\left(l_x/2-K_2(a/2)\right)+a_{i-1}^{2,0}(a/2)^2, \quad (8c)$$

$$\gamma_y=T_{i,0}^yK_1(b/2)-Q_{i,0}^y\frac{K_2(b/2)}{k^y}+\frac{2a_{i-1}^{0,2}}{b_y^2}\left(1-K_1(b/2)\right)$$
$$+a_{i-1}^{0,1}\left(b/2-K_2(b/2)\right)+a_{i-1}^{0,2}(b/2)^2. \quad (8d)$$

So, formally the constants A_i and $T_{c,i}$ are the linear combinations of the initial parameters $T_{i,0}^x$, $Q_{i,0}^x$ – at the left inlet left side, and $T_{i,0}^y$ $Q_{i,0}^y$ – at the inlet lower side of the element; as well as the coefficients $a_{i-1}^{k,m}$ (2e), which are "inherited" from the previous time step analysis.

Thus, substituting the expressions (8a) and (8b) again into equations (6c), (6d) and (7a), (7b) we get the complete solution for all these four functions withing the considered element:

$$\begin{pmatrix}T_i^x(x)\\Q_i^x(x)\\T_i^y(y)\\Q_i^y(y)\end{pmatrix}=\begin{Bmatrix}\alpha_{1,1}(x); & \alpha_{1,2}(x); & \alpha_{1,3}(x); & \alpha_{1,4}(x)\\\alpha_{2,1}(x); & \alpha_{2,2}(x); & \alpha_{2,3}(x); & \alpha_{2,4}(x)\\\alpha_{3,1}(y); & \alpha_{3,2}(y); & \alpha_{3,3}(y); & \alpha_{3,4}(y)\\\alpha_{4,1}(y); & \alpha_{4,2}(y); & \alpha_{4,3}(y); & \alpha_{4,4}(y)\end{Bmatrix}\begin{pmatrix}T_{i,0}^x\\Q_{i,0}^x\\T_{i,0}^y\\Q_{i,0}^y\end{pmatrix}$$
$$+\sum_{k,m}a_{i-1}^{k,m}\begin{pmatrix}b_1^{k,m}(x)\\b_2^{k,m}(x)\\b_3^{k,m}(y)\\b_4^{k,m}(y)\end{pmatrix}. \quad (9a)$$

The derived matrix equation is called as a Connection equation. Substituting instead x and y their values at the outlet border (side) of the element, we get the relation between the inlet values of the main parameters and the outlet values:

$$\begin{pmatrix}T_{i,e}^x\\Q_{i,e}^x\\T_{i,e}^y\\Q_{i,e}^y\end{pmatrix}=\begin{Bmatrix}\alpha_{1,1}(a); & \alpha_{1,2}(a); & \alpha_{1,3}(a); & \alpha_{1,4}(a)\\\alpha_{2,1}(a); & \alpha_{2,2}(a); & \alpha_{2,3}(a); & \alpha_{2,4}(a)\\\alpha_{3,1}(b); & \alpha_{3,2}(b); & \alpha_{3,3}(b); & \alpha_{3,4}(b)\\\alpha_{4,1}(b); & \alpha_{4,2}(b); & \alpha_{4,3}(b); & \alpha_{4,4}(b)\end{Bmatrix}\begin{pmatrix}T_{i,0}^x\\Q_{i,0}^x\\T_{i,0}^y\\Q_{i,0}^y\end{pmatrix}$$

$$+\sum_{k,m} a_{i-1}^{k,m} \begin{pmatrix} b_1^{k,m}(a) \\ b_2^{k,m}(a) \\ b_3^{k,m}(b) \\ b_4^{k,m}(b) \end{pmatrix}. \qquad (9b)$$

Where the special designation for main parameters at the outlet sides are introduced $T_{i,e}^x = T_i^x(a)$, $Q_{i,e}^x = Q_i^x(a)$, $T_{i,e}^y = T_i^y(b)$, $Q_{i,e}^y = Q_i^y(b)$, Fig 1.

Now it is easy to formulate the general algorithm of solution, which is identical to this one proposed in our works [24], [25] except the number of main parameters of the problem (number of unknowns). So, repeat it in short. First of all, we break out the whole structure on number of elements, say, G. Then introduce 8 unknowns for each element, which are 2 parameters on each of 4 sides of element. At the whole, there $G \cdot 8 = 8G$ unknowns.

This number should be equal to the number of equations. Count them. At each border between two elements there are two continuity equations (Conjugation), which state that temperatures and fluxes at both conjugated sides are the same. Formally, this means that for each border provides 1 equation for each element. In case if the side of the element is the boundary of the whole body, then 1 boundary condition should be formulated for it. So, the general rule that 1 side gives 1 equation still hold. So, for each element we have 4 Conjugation (boundary) equations, at the whole for the structure they are equal to $4G$. Other $4G$ are derived as 4 Connection equations (9b) for each element. So, the number of equations and number of unknowns does coincide, and algorithm of solution reduces to a) proper meshing the structure; b) proper organization of bypass through the structure and numbering the elements; c) proper numbering the unknowns; d) compilation and solution of the matrix equation; e) presentation of results inside each element. All these essential steps were explained in our works [24], [25].

3. Presentation of the solution from the previous moment of time.

This part of work relates to the treatment of the obtained solution at the i time step and the preparation of data for the problem solution at the next $i+1$ time step. So, the task is to get the presentation of the temperature in a form:

$$T_i(x,y) = a_i^{0,0} + a_i^{1,0}x + a_i^{0,1}y + a_i^{2,0}x^2 + a_i^{0,2}y^2. \qquad (10a)$$

With some small amendment we repeat here the procedure given in [29], [30]. First of all, note that availability of solution at the time i means that $T_{i,0}^x, Q_{i,0}^x$ and $T_{i,0}^y, Q_{i,0}^y$ are already known, and according to relations (6c) and (7a) the distribution of the temperatures can be presented as:

$$T_i^x(x) = \beta_{i,1}^x K_1(x) + \beta_{i,2}^x K_2(x) + \tilde{T}_i^x(x), \qquad (10b)$$

$$T_i^y(y) = \beta_{i,1}^y K_1(y) + \beta_{i,2}^y K_2(y) + \tilde{T}_i^y(y). \qquad (10c)$$

Where $\beta_{i,\mathrm{j}}^x$, and $\beta_{i,\mathrm{j}}^y$ are known coefficients and $\tilde{T}_i^x(x)$ and $\tilde{T}_i^y(y)$ are known polynomials of the second degree, see for example (6a). The next step is the expansion of the functions $K_{1,\mathrm{or},2}(x\, or\, y)$ into polynomial series of the second degree [29]. The integrally averaged procedure is suggested to use. According to it we present, for example, $K_1(x)$ as:

$$K_1(x) = ch(xb_x) = f_0^x + f_1^x x + f_2^x x^2. \qquad (11a)$$

Then consequently multiplying both sides of (11a) by 1, x, and x^2, and integrating it over x from $x=0$, to $x=a$ we get the system of three equations which gives the values of coefficients f_0^x, f_1^x, f_2^x as the function of value (ab_x). In similar way we can get the expansion of $K_2(x)$ as:

$$K_1(x) = \frac{\mathrm{sh}(xb_x)}{b_x} = g_0^x + g_1^x x + g_2^x x^2. \qquad (11b)$$

Thus, it is easy to see that accounting for (6a), (11a) and (11b) the function $T_i^x(x)$ can be presented as second-degree polynomial:

$$T_i^x(x) = T_{i,x}^0 + T_{i,x}^1 x + T_{i,x}^2 x^2. \qquad (12a)$$

Similarly the distribution of the temperature in y direction can be written as:

$$T_i^y(y) = T_{i,y}^0 + T_{i,y}^1 y + T_{i,y}^2 y^2. \qquad (12b)$$

The last step we should do, and which was absent in 1D wave propagation problem [29] is to merge two independent solutions into a general one. Recall the formal presentation of temperature within the element (2c). Find the temperature in the center of the element according to each of two presentations (12a) and (12b):

$$\begin{aligned} T_{i,c}^x &= T_{i,x}^0 + T_{i,x}^1 a/2 + T_{i,x}^2 (a/2)^2; \\ T_{i,c}^y &= T_{i,y}^0 + T_{i,y}^1 b/2 + T_{i,b}^2 (b/2)^2. \end{aligned} \qquad (12c)$$

Then consider the conventional temperature at the center from the previous time stem as the semi-sum of $T_{i,c}^x$ and $T_{i,c}^y$. Apply the general rule of the temperature presentation (2c), so we get:

$$T_i(x,y) = \frac{T_{i,x}^0 + T_{i,y}^0}{2} + T_{i,x}^1\left(x - \frac{a}{4}\right) + T_{i,x}^2\left(x^2 - \frac{a^2}{8}\right)$$
$$+ T_{i,y}^1\left(y - \frac{b}{4}\right) + T_{i,y}^2\left(y^2 - \frac{b^2}{8}\right). \qquad (12d)$$

The comparison of (2e) with (12d) allows to get the required values of $a_i^{m,k}$. So, all data are available to proceed to the next time iteration.

4. Example of the solution.

4.1. Problem statement

Consider the rectangular plate $0 \le x \le L_x$, $0 \le y \le L_y$, which sides are the same $L_x = L_y$ and equal to 5. Take that all physical coefficients are constants and equal to 1, so $c = \rho = k^x = k^y = 1$. Assume that heat transfer is absent at all plate sides, thus:

$$\frac{\partial T}{\partial x}(x=0,y,t)=0\,,\quad \frac{\partial T}{\partial x}(x=L_x,y,t)=0\,,\qquad (13a)$$

$$\frac{\partial T}{\partial y}(x,y=0,t)=0\,,\quad \frac{\partial T}{\partial y}(x,y=L_y,t)=0\,.\qquad (13b)$$

Consider that initial temperature distribution, $T_0(x,y,t=0)$ is given by two dimensional Gauss function with the following parameters $\mu_x = 2.5$, $\mu_y = 2.5$, and $\sigma^2{}_x = \sigma^2{}_y = 1$:

$$T_0(x,y)=\exp\left(-\frac{1}{2}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right)\right).\qquad (13c)$$

The initial temperature distribution is shown on Fig 2

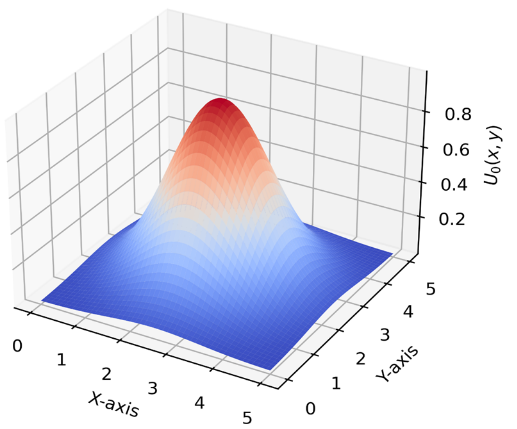

Fig 2. Initial temperature distribution, $t = 0$

4.2. Analytical solution

This task allows the exact solution by classical Fourier method of separation of variables. So, the looking for temperature is presented as:

$$T(x,y,t)=X(x)Y(y)U(t)\qquad (14a)$$

The general solution with accounting for the simple boundary conditions (13a) and (13b) can be presented as [8]:

$$T(x,y,t)=\sum_{n=0}^{N}\sum_{m=0}^{M}C_{nm}\cos\frac{n\pi x}{L_x}\cos\frac{m\pi y}{L_y}$$

$$\times\exp\left(-\frac{\pi^2}{c\rho}\left(k^x\left(\frac{n}{L_x}\right)^2+k^y\left(\frac{m}{L_y}\right)^2\right)t\right).\qquad (14b)$$

At initial (zero) time the temperature is given by the following expression:

$$T_0(x,y)=\sum_{n=0}^{N}\sum_{m=0}^{M}C_{nm}\cos\frac{n\pi x}{L_x}\cos\frac{m\pi y}{L_y}.\qquad (14c)$$

To find the unknown coefficients C_{nm} we need to consequently multiply both sides of (14c) for any combinations of integer (n,m) on the shape functions $\cos\frac{n\pi x}{L_x}\cos\frac{m\pi y}{L_y}$, and integrate them over the plate area. Note that left side of (14c) is equal to initial distribution (13c). This allows to find C_{nm} and then apply (14b) for the temperature determination at any moment of time t. The right hand of (14c) is integrated analytically. To perform the integration of left side take 450×450 evenly distributed point within the plate area and determine C_{nm} for all $0 \le n \le 30 = N$ and $0 \le m \le 30 = M$, at the whole 900 coefficients are determined.

The demonstration of the accuracy of the Fourier presentation is given in Table 1. Here in initial time the theoretically calculated temperatures (right side of (14c)) are compared with initial distribution (13c) for some chosen plate points. As we see there is a very good correspondence. Some difference is related with absence of higher terms of Fourier expansion. But these higher terms according to (14b) decays very quickly with time. This will be demonstrated later. So, for any intermediate moments of time the 30×30 Fourier solution can be considered as exact one.

Table 1. Comparison between initial given temperature and calculated one by Fourier method with application of 450×450 integration points and 30×30 analytical terms

$(x,y)\backslash T$	Exact values	Analytical expansion	Difference
(2.5,2.5)	1.0	0.999739	0.000261
(3.5,2.5)	0.606531	0.606317	0.000214
(4.5,2.5)	0.135335	0.135119	0.000216
(3.5,3.5)	0.367879	0.367716	0.000163
(4.5,4.5)	0.018316	0.018262	0.000054

The availability of analytical solution allows to build the dependence of temperature with time, for example, in

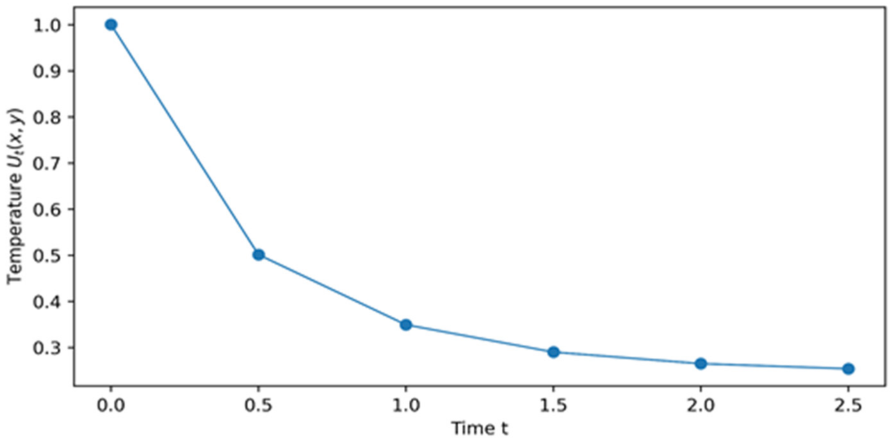

Fig 3. The theoretical distribution of the temperature in the central point, $x = 2.5,\ y = 2.5$

Table 2. The analytically calculated temperature for two different meshing and number of terms: a) 450×450 points, 30×30 terms, b) 300×300 points and 20×20 terms

$(x, y)\backslash T$	0.25	0.5	0.75	1.0	1.5	2.5	5
(2.5,2.5)	0.666695	0.501525	0.407017	0.349554	0.290220	0.254108	0.245295
(3.5,2.5)	0.478584	0.396065	0.343811	0.310239	0.273981	0.250976	0.245236
(4.5,2.5)	0.198708	0.229727	0.242338	0.246779	0.247710	0.245908	0.245140
(3.5,3.5)	0.343550	0.312781	0.290420	0.275346	0.258650	0.247883	0.245177
(4.5,4.5)	0.059225	0.105228	0.144288	0.174222	0.211427	0.237973	0.244985

the central point of plate $(x = 2.5, y = 2.5)$, Fig 3. Evidently, the temperature changes most drastically at the moment of time $t = 0.25$, so it will be mostly taken for comparison with our direct time integration procedure in MMS.

Theoretical results of calculation for chosen plate points at the intermediate moments of time are presented in Table 2. They are absolutely the same at application two different number of integration points and number of expansion terms. So, we can state that analytical results are exact for given number of significant digits. So, the comparison with them will allow to judge about the accuracy of our method.

4.3. Our results and comparison

Analyze the results obtained by our method. First of all, note that our method employs the functions of the form $ch(xb_x)$. This function become very large within the element when the value of ab_x exceed some number. So, the machine mistake might occur. This was already analyzed in our work [29], where it was suggested to restrict the upper value of ab_x, by, say number of 10. Recalling the expression for b_x we can write

$$D = \sqrt{\frac{c\rho}{\Delta k}}a = \frac{a}{\sqrt{\Delta}} \le D_0 \tag{15a}$$

Where D_0 is the maximum allowable argument of exponential function in the numerical calculation due to machine error. This criterion establishes the requirement to the minimum time step Δ when the element size a is already chosen:

$$\Delta \ge \left(\frac{a}{D_0}\right)^2. \tag{15b}$$

To illustrate the above consideration, perform the calculations for two points of plate at time $t = 0.25$ for different time steps Δ with application of 15×15 meshing. The results are given in Table 3. Evidently, the accuracy is becoming better with taking less time step. But when the time step become very small the argument of exponential function become very large, so the solution diverges. As we see from Table 3 it is expedient to restrict the values of D_0 by value of 10. So, in all subsequent analysis we will follow this restriction.

Perform the similar analysis for two different more refined meshing, 45×45 and 65×65. The results are shown in Tables 4 and 5. Evidently, the accuracy become better with refining the space meshing as well with decreasing the time step. Furthermore, namely the decreasing the length the element allows to decrease the time step.

Table 3. Calculated temperature at time 0.25 for different time step Δ with application of 15×15 meshing

D		33.3334	14.9071	10.5409	6.6667	4.7140	3.3334	2.1082	1.4907	0.6667
$(x,y)\backslash\Delta$	Fourier	0.0001	0.0005	0.001	0.0025	0.005	0.01	0.025	0.05	0.25
(2.5,2.5)	0.666695	1.473361	0.667471	0.667621	0.668067	0.668805	0.670266	0.674543	0.681338	0.724307
(4.5,2.5)	0.198708	0.52406	0.198216	0.198180	0.198073	0.197894	0.197542	0.196520	0.194924	0.185486

Table 4. Calculated temperature at time 0.25 for different time step Δ with application of 45×45 meshing

D		11.1111	4.9690	3.5136	2.2222	1.5713	1.1111	0.7027	0.4969	0.2222
$(x,y)\backslash\Delta$	Fourier	0.0001	0.0005	0.001	0.0025	0.005	0.01	0.025	0.05	0.25
(2.5,2.5)	0.666695	0.666800	0.666919	0.667068	0.667514	0.668254	0.669720	0.674012	0.680831	0.723948
(4.5,2.5)	0.198708	0.198647	0.198618	0.198581	0.198472	0.198292	0.197935	0.196901	0.195286	0.185743

Table 5. Calculated temperature at time 0.25 for different time step Δ with application of 65×65 meshing

D		7.6923	3.4401	2.4325	1.5385	1.0879	0.7692	0.4865	0.3440	0.1538
$(x,y)\backslash\Delta$	Fourier	0.0001	0.0005	0.001	0.0025	0.005	0.01	0.025	0.05	0.25
(2.5,2.5)	0.666695	0.666761	0.666880	0.667029	0.667475	0.668215	0.669681	0.673974	0.680795	0.723920
(4.5,2.5)	0.198708	0.198675	0.198645	0.198609	0.198500	0.198319	0.197962	0.196927	0.195311	0.185762

So, the question is which parameter of integration – the spatial or temporal ones have the larger influence on the accuracy? In work [29] for 1D problem the answer was that the time step predetermines the accuracy of the numerical scheme, while significance of the space meshing is in controlling the allowable value of D (argument of exponential function).

To give the answer introduce the measure of accuracy, M_ε, by the following expression:

$$M_\varepsilon = log\left|T_{MMS} - T_{exact}\right| . \tag{16a}$$

This measure will be presented in dependance with logarithm of inverse of applied time step, Δ_ε

$$\Delta_\varepsilon = log\left(1/\Delta\right) . \tag{16b}$$

The generalized graph of dependence of accuracy from the time step is shown on Fig 4. To construct it we apply the different combinations of spatial meshing and time steps.

As in works [29], [30] we can state that time step is the main parameter which control the accuracy. As to spatial meshing the linear size of element should be small enough to exclude the mashing error of calculation of exponential functions at large arguments.

It is of interest to compare the accuracy of our MMS with traditional FEM results. Fig 5 shows the lower envelop of our results (for very fine spatial mesh) as well as the results obtained by ABAQUS with use of CPS4T element. Note that numbers in parentless show the number of elements along one side. As we see our results give better accuracy especially for small time steps. All above show the high accuracy of the method.

One another question about the accuracy of our method remains unanswered. It might be argued that the demonstrated accuracy was attained only in two points and only in one moment of time, $t = 0.25$. To answer this question, we recalculate by our MMS the values of temperature in the same time and space points as in Table 2. Adopt the intermediate space meshing 65×65 and time step $\Delta = 0.001$ and present the difference of calculated and exact temperatures in Table 6. So, the very good correspondence is achieved for all space points and the moments of time. Other important peculiarity of the method is that results at large time converge to the correct value of temperature. This means that occurs no numerical dissipation of energy.

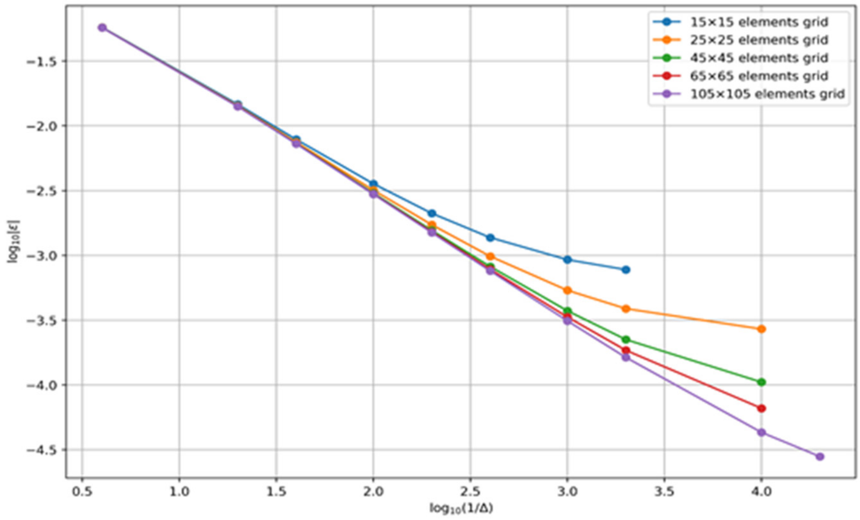

Fig. 4: Dependence of the measure of accuracy, M_{ε} , with logarithm of inverse of time step, Δ_{ε} for different spatial meshing

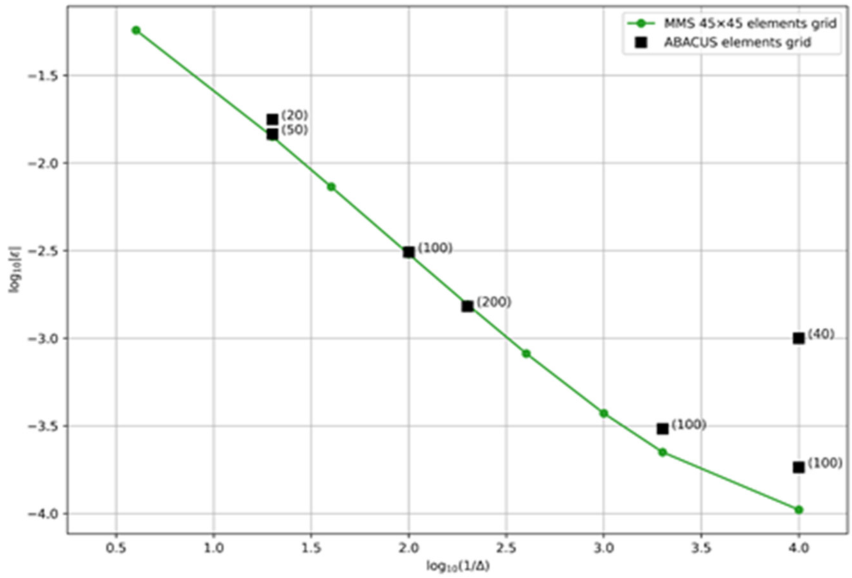

Fig 5. Comparison accuracy attained by MMS and by ABAQUS

Table 6. Difference of results obtained by MMS with exact ones for different time and space points

$(x,y)\backslash T$	0.25	0.5	0.75	1.0	1.5	2.5	5
(2.5,2.5)	0.334	0.296	0.233	0.177	0.099	0.029	0.001
(3.5,2.5)	0.102	0.127	0.115	0.096	0.058	0.019	0.001
(4.5,2.5)	0.099	0.093	0.062	0.034	0.006	0.002	0
(3.5,3.5)	0.002	0.039	0.046	0.041	0.026	0.008	0
(4.5,4.5)	0.023	0.013	0.044	0.058	0.053	0.022	0.001

Conclusion

This work expands the application of the original method of matched section to the heat conduction problem. The gist of the method is an approximate solution of the governing differential equations, so the solution is presented in strong form rather than in weak one as in traditional FEM. In first time it considers the transient 2D problem by original implicit time integration method, where the element interpolation functions explicitly depend on the time step. The task for redistribution of the temperature within the rectangular plate is considered in detail. It is shown that method is stable for all time step irrespective the spatial mesh. The only restriction is related with that argument in exponential interpolation function (inner solution) is inversely proportional to square root of time step. So, at very small time step the interpolation function became incontrollable large due to machine errors and results became incorrect. It is shown that main parameter in MMS which control accuracy is a time step. As to spatial meshing the linear size of element should be small enough to exclude the mashing error of calculation of exponential functions at large arguments.

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Неявне пряме інтегрування за часом задачі теплопровідності в методі узгоджених перерізів

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Анотація. *Стаття присвячена подальшому розвитку методу узгоджених перерізів як нового напряму методу скінченних елементів у застосуванні до перехідної 2D температурної задачі. Основна відмінність МУП від традиційного МСЕ полягає в тому, що спряження здійснюється між сусідніми перерізами, а не у вузлах елементів. Важливою особливістю методу є те, що він базується на наближеному розв'язку в сильній формі визначальних диференціальних рівнянь, які тут називаються рівняннями зв'язку. Передбачається, що для кожного малого прямокутного елемента двовимірну задачу можна розглядати як комбінацію двох одновимірних задач – одна з них залежить від x, а інша – від y. Кожна задача характеризується двома функціями – температурою, та тепловим потоком. У практичній реалізації для прямокутних скінченних елементів метод зводиться до визначення восьми невідомих для кожного елемента – по два невідомих з кожного боку, які пов'язані рівняннями зв'язку, та вимогою неперервності температури в центрі елемента. Іншою важливою особливістю роботи є реалізація оригінальної неявної схеми інтегрування за часом, де крок за часом стає параметром функції форми в межах елемента, тобто визначає поведінку рівнянь зв'язку. Цей метод був вперше запропонований автором для ряду одномірних задач, а тут вперше застосований для двовимірних задач. Ряд тестів для прямокутної пластини демонструє чудові властивості цієї "вбудованої" схеми інтегрування за часом щодо стійкості, точності та відсутності будь-яких обмежень щодо збільшення кроку за часом.*

Ключові слова: *Метод узгоджених перерізів, неявне інтегрування за часом, залежні від кроку за часом функції форми, прямокутна пластина, перехідна температура.*